
SHORT NOTE

Recommendation on the standard definition of the discrete Fourier transform

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The forward and inverse discrete Fourier transforms (DFT and IDFT) are processes of converting one sequence of numbers to another sequence of numbers and converting it back to the original according to its conversion rule. The two sequences look different, but since they are completely reversible, they may be two different views or expressions of one object. In the field of signal processing, one sequence is a (mostly real-valued) time-dependent signal and another sequence is a frequency-dependent spectrum. If one surveys books covering the Fourier analysis, he/she will find that there are a couple of variations of definitions. Those are:

Definition I [1-3]

$$X_I(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \leftrightarrow x(n) = \sum_{k=0}^{N-1} X_I(k)e^{j2\pi kn/N}$$

Definition II [4-7]

$$X_{II}(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ \leftrightarrow x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_{II}(k)e^{j2\pi kn/N}$$

Definition III [6,7]

$$X_{III}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ \leftrightarrow x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_{III}(k)e^{j2\pi kn/N}$$

One author gives one of the above definitions in his book and comments generously that there are other definitions, which are basically the same. Since the differences are only the coefficients $1/N$ or $1/\sqrt{N}$ in front of the summations, any definition can be used without problem if only the temporal and/or spectral profiles matter. However, for application-oriented people, it seems very natural to have the following questions: "Why does a signal have different spectral magnitudes? Do they give similar results as the continuous Fourier transforms, or which of them is more analogous to the continuous transforms? Is the energy (or power) conserved?"

In fact, on my personal computer, there are three DFT/IDFT programs written in Visual Basic, Java, and MATLAB languages, and all of them give different spectral magnitudes. It is troublesome to remember which definition is used in each of them. Also there is a chance that results are interpreted with a wrong coefficient(s).

Considering the above, I would like to comment on my preference among the three definitions. Since Definitions I and II are more common than Definition III, the first two will be discussed. Similar to the discrete transform, there are several ways of defining the continuous Fourier transform. However, in order to make the situation simpler, only the following definition will be considered:

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft} dt \leftrightarrow x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft} df$$

2. Analogy to the Fourier coefficient expansion of a periodic signal

The definitions of discrete transforms may be derived independently from continuous transforms. However, since the latter has a much longer history, the analogy may make the understanding of the discrete transform easier. It is known that the discrete Fourier transform is periodic in the two domains (time and frequency, for example); the analogy may hold more for cases with continuous periodic signals. Indeed, Definition I can be obtained from the direct analogy to the Fourier series coefficients of the continuous system. For a periodic signal $x(t)$ with period T , the signal can be expressed by the complex Fourier coefficients $X(k)$:

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)e^{j2\pi kt/T} \quad (1)$$

$$X(k) = \frac{1}{T} \int_0^T x(t)e^{-j2\pi kt/T} dt \quad (2)$$

If the signal $x(t)$ in the period T is sampled at equally spaced N points ($T = N\Delta t$), Eqs. (1) and (2) can be approximated, respectively, by

$$x(n) = \sum_{k=0}^{N-1} X_I(k)e^{j2\pi kn\Delta t/N\Delta t} = \sum_{k=0}^{N-1} X_I(k)e^{j2\pi kn/N} \quad (3)$$

$$X_I(k) = \frac{1}{N\Delta t} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \Delta t = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad (4)$$

Equations (3) and (4) are identical to the discrete Fourier transforms given by Definition I. From the analogy, each coefficient given by Eq. (4) can be understood as the complex amplitude of the k -th harmonic of a periodic signal.

3. Analogy to the continuous Fourier transform**3.1. DC signal**

If one of the definitions gives values closer to values expected from the continuous Fourier transform, that definition may be more preferable than others. The continuous Fourier transform of a constant 1 is given by $\delta(f)$. It may not

be easy to understand the delta function. However, since it is natural to consider that the 1 V DC signal has a spectrum (amplitude) of 1 V at frequency 0, $\delta(f)$ may be interpreted as “amplitude of 1 at frequency 0.”

For the case of the DC signal, Definitions I and II give values $X_I(0) = 1$, $X_I(k) = 0$ ($k \neq 0$), and $X_{II}(0) = N$, $X_{II}(k) = 0$ ($k \neq 0$), respectively. It seems that the results of Definition I is more easily accepted considering the analogy to the continuous system.

3.2. Cosine signal

The continuous Fourier transform of $\cos(2\pi f_0 t)$ is given by $[\delta(f - f_0) + \delta(f + f_0)]/2$. The same understanding as the case of the DC signal is that the cosine signal has the components with amplitude 1/2 at frequencies $\pm f_0$.

Let's consider a case with $f_0 = 1$ (Hz), and the signal is sampled with sampling frequency of 8 (Hz), and $N = 8$. $N = 8$ covers one period (1 s) of the signal. In this case, Definitions I and II give the values, $X_I(\pm 1) = 1/2$, $X_I(k) = 0$ ($k \neq \pm 1$), and $X_{II}(\pm 1) = 4$, $X_{II}(k) = 0$ ($k \neq \pm 1$), respectively. It seems that Definition I again gives a result that fits better with the continuous Fourier transform.

4. Power equality

From Definitions I and II, the following equations are obtained.

$$\frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |X_I(k)|^2 \quad (5)$$

$$\frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N^2} \sum_{k=0}^{N-1} |X_{II}(k)|^2 = \sum_{k=0}^{N-1} |X_{II}(k)|^2 / N^2 \quad (6)$$

Since the instantaneous power of signal $x(n)$ is $|x(n)|^2$, the left-hand side of Eq. (1) (and (2)) is the average power of the signal. Since $|X(k)|^2$ is normally defined as the power spectrum, Definition I gives the power equality that “the average power in the time domain equals the sum of the power spectrum in the frequency domain.” If Definition II is used, the power equality must be stated that “the average power equals the sum of the power spectrum divided by N squared.” It is obvious to me that the former is much

simpler in expressing the power equality and is more preferable.

Let's check whether Eq. (5) is valid or not for the two cases discussed above. In the case of the DC signal, the average power is 1, which equals $\sum_{k=0}^{N-1} |X(k)|^2 = |X(0)|^2 = 1$. In the case of the cosine signal, $\{\sum_{n=0}^{N-1} |x(n)|^2\}/N$ is 1/2, which is also equal to $\sum_{k=0}^{N-1} |X(k)|^2 = |X(+1)|^2 + |X(-1)|^2 = 1/2$.

5. Conclusions

There are a couple of definitions of the discrete Fourier transform and accordingly different versions of algorithms are available through the internet or other media. If the absolute values of the transformation matter in a particular application(s), one must always keep in mind which definition is being used in the program. This is a headache and sometimes may cause problems. From the discussions above, I recommend Definition I as the standard form of the forward and inverse discrete Fourier transforms.

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