

PAPER

Instantaneous frequencies of signals obtained by the analytic signal method

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Abstract: In this paper, the instantaneous frequency is defined as the one obtained by converting a real time signal to a complex analytic signal and by differentiating the time-dependent phase with respect to time. Theoretical expressions of instantaneous frequencies for signals given as combinations of sinusoidal components are presented. Those are compared to the results obtained by numerical methods using the discrete Fourier transform in order to confirm the validity of the expressions and to check accuracies of the numerical methods. A reason for the existence of negative instantaneous frequencies is given by a vector representation of signal components. The instantaneous frequencies of frequency- and amplitude-modulated signals are also discussed. For a periodic sinusoidal burst signal, it was found that the instantaneous frequency stays zero during the period when the signal is zero and takes a value equal to twice that of the sinusoid at the onset of the signal.

Keywords: Signal analysis, Instantaneous frequency, Analytic signal

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1. INTRODUCTION

There are occasions when time-dependent frequencies of signals are needed. Music transcription is one such case where time-dependent fundamental frequencies of various musical instruments played together must be found. Another example is the pitch detection of speech. The fundamental frequencies of syllables (pitches) are required for further analysis of spoken words in order to provide prosodic information in speech synthesis/recognition. Various methods are used for time-dependent frequency analysis. The auto-correlation method is frequently used in speech analysis [1] as well as the zero-crossing method [2]. Another method is to apply the short-time Fourier transform to the signal and find instantaneous frequencies at different times and frequencies of analysis [3]. A similar method is to apply time-frequency analysis methods such as the Wigner distribution [4,5] and wavelet analysis [6].

Actually, any method may be qualified as a method for obtaining the "instantaneous frequency" if resulting frequencies are obtained with enough time resolution for the analysis of a phenomenon of interest. However, "instantaneous frequency" is quite often used in a much narrower

sense [7]. In reference [7], the instantaneous frequency is obtained in the following way: a time signal is converted into an "analytic signal" that contains only positive frequency spectra of the original signal, then the instantaneous frequency (denoted "IF" hereafter) is obtained by the time-derivative of the instantaneous phase of the analytic signal. In some cases, the IF is very useful to clarify differences of signals even if the time waveforms look very similar to each other [8]. It can also be used for the analysis of vibratos, e.g. of violin tones [9].

Then, the question "What is the IF?" arises. For example, what is the IF of a signal composed of two sinusoids with arbitrary ratios of frequencies and amplitudes? Or, what is the IF of a signal that is zero for $t < 0$ and equal to a hemi-infinite sine function starting from $t = 0$? In this paper, IFs of several signals with tonal components are theoretically evaluated in order to better understand properties of IFs. In cases when negative IFs are observed, a reason for it will be given. The theoretical expressions are compared with numerical evaluations. Agreements between those results will help confirm the validity of the theory.

2. INSTANTANEOUS FREQUENCY

The IF used in this paper is defined here as the frequency obtained by the following steps [7]:

- (1) Apply the Fourier transform to a real time-dependent signal.
- (2) Set the negative and DC components equal to zero in order to make the signal "analytic."
- (3) Apply a band-pass filter, if necessary, in order to analyze a band-limited signal.
- (4) Apply the inverse Fourier transform to the analytic signal in the frequency domain.
- (5) Obtain the instantaneous phase from its real and imaginary components.
- (6) Differentiate the instantaneous phase with respect to time and divide by 2π .

A simple example is given here. Let the signal $x(t)$ be a cosine function:

$$x(t) = A \cos(\omega_0 t) = A[e^{j\omega_0 t} + e^{-j\omega_0 t}]/2 \quad (1)$$

Then, we have the following equations corresponding to the steps shown above.

$$X(\omega) = A[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]/2 \quad (2)$$

$$X_a(\omega) = A\delta(\omega - \omega_0)/2 \quad (3)$$

$$x_a(t) = (A/2)e^{j\omega_0 t} \quad (4)$$

$$\phi_1(t) = \omega_0 t \quad (5)$$

$$f_1(t) = \frac{1}{2\pi} \frac{d\phi_1(t)}{dt} = f_0 \quad (6)$$

where $X(\omega)$, $X_a(\omega)$, $x_a(t)$, $\phi_1(t)$, $f_1(t)$ are the Fourier spectrum, analytic Fourier spectrum, analytic time signal, instantaneous phase, and IF, respectively. The IF in this case is a constant, f_0 , as expected.

This method is quite straightforward and can be applied to any finite discrete signal in order to numerically evaluate the IF (see Sect. 4).

3. IFs OF VARIOUS SIGNALS

In this section, IFs of some typical signals such as a

$$\omega_1(t) = \frac{a_1^2 \omega_1 + a_2^2 \omega_2 + a_1 a_2 (\omega_1 + \omega_2) \cos[(\omega_1 - \omega_2)t + (\theta_1 - \theta_2)]}{a_1^2 + a_2^2 + 2a_1 a_2 \cos[(\omega_1 - \omega_2)t + (\theta_1 - \theta_2)]} \quad (12)$$

If $a_1 = a_2$, Eq. (12) is equal to a constant:

$$\omega_1(t) = \frac{\omega_1 + \omega_2}{2} \quad (13)$$

giving a simple and reasonable result.

Times at which maximum and minimum instantaneous angular frequencies occur are obtained by differentiating Eq. (12) with respect to time and letting the result equal to zero.

$$\frac{\partial \omega_1(t)}{\partial t} = \frac{a_1 a_2 (a_1^2 - a_2^2) (\omega_1 + \omega_2) (\omega_1 - \omega_2) \sin(\omega_1 - \omega_2)t}{[a_1^2 + a_2^2 + 2a_1 a_2 \cos(\omega_1 - \omega_2)t]^2} = 0 \quad (14)$$

signal with two frequency components, a signal with multiple frequency components, frequency-modulated (FM) and amplitude-modulated (AM) signals, and a periodic sinusoidal burst signal are treated. Expressions of IFs of these signals are theoretically available and will be used to evaluate the accuracies of numerical methods.

IFs of signals with two frequency components are discussed in reference [4]. Those of FM and AM signals are also discussed in more general cases in reference [4]. In this paper, details of the derivation of theoretical expressions for IFs of these signals are given.

3.1. Signals with Two Frequency Components

The IF of a signal with two frequency components is discussed in reference [4]. In this section, details of the derivation of the theoretical expression and properties of the IF are described.

Let's consider a signal with two frequency components:

$$x(t) = a_1 \cos(\omega_1 t + \theta_1) + a_2 \cos(\omega_2 t + \theta_2) \quad (7)$$

Then the analytic time signal is given by

$$x_a(t) = \{[a_1 \cos(\omega_1 t + \theta_1) + a_2 \cos(\omega_2 t + \theta_2)] + j[a_1 \sin(\omega_1 t + \theta_1) + a_2 \sin(\omega_2 t + \theta_2)]\}/2 \quad (8)$$

The instantaneous phase follows as

$$\phi_1(t) = \tan^{-1} \left[\frac{a_1 \sin(\omega_1 t + \theta_1) + a_2 \sin(\omega_2 t + \theta_2)}{a_1 \cos(\omega_1 t + \theta_1) + a_2 \cos(\omega_2 t + \theta_2)} \right] \quad (9)$$

If we let

$$y = \frac{a_1 \sin(\omega_1 t + \theta_1) + a_2 \sin(\omega_2 t + \theta_2)}{a_1 \cos(\omega_1 t + \theta_1) + a_2 \cos(\omega_2 t + \theta_2)} \quad (10)$$

we can obtain the instantaneous angular frequency by

$$\omega_1(t) = \frac{d\phi_1(t)}{dt} = \left(\frac{d\phi_1}{dy} \frac{dy}{dt} \right) = \frac{1}{1+y^2} \frac{dy}{dt} \quad (11)$$

After a rather laborious process, we obtain the equation for the instantaneous angular frequency:

Thus, the maximum and minimum instantaneous angular frequencies exist at times $t = n\pi/(\omega_1 - \omega_2)$ (n : integer) and the instantaneous angular frequency takes the following values:

$$\omega_i = \begin{cases} (a_1\omega_1 + a_2\omega_2)/(a_1 + a_2) & n: \text{even} \\ (a_1\omega_1 - a_2\omega_2)/(a_1 - a_2) & n: \text{odd} \end{cases} \quad (15)$$

If, for example, $(a_1/a_2) > 1$ and $(a_1/a_2) < (\omega_2/\omega_1)$ are satisfied, the angular frequency can take negative values.

3.2. Signals with Multiple Frequency Components

For a signal with M frequency components, we can follow exactly the same procedure taken in Sect. 3.1, resulting in a formula shown below:

$$\omega_i(t) = \frac{\sum_{m=1}^M a_m^2 \omega_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M a_m a_n (\omega_m + \omega_n) \cos[(\omega_m - \omega_n)t + (\theta_m - \theta_n)]}{\sum_{m=1}^M a_m^2 + \sum_{m=1}^{M-1} \sum_{n=m+1}^M 2a_m a_n \cos[(\omega_m - \omega_n)t + (\theta_m - \theta_n)]} \quad (16)$$

3. Frequency- and Amplitude-Modulated Signals

IFs of frequency-modulated (FM) and amplitude-modulated (AM) signals are also discussed in reference [4]. In this paper, IFs for the cases when the carrier and the modulation frequencies are both single sinusoids are discussed.

An FM signal with a constant amplitude is expressed by

$$x(t) = \sin \left[\omega_c t - \frac{\omega_d}{\omega_m} \cos(\omega_m t) \right] \quad (17)$$

where ω_c and ω_m are the carrier and modulation frequencies and ω_d/ω_m ($=\beta$) is the modulation index. Equation (17) can be rewritten as

$$x(t) = [e^{j(\omega_c t - \beta \cos(\omega_m t))} - e^{-j(\omega_c t - \beta \cos(\omega_m t))}] / 2j \quad (18)$$

Using the Jacobi's expansion formulae in terms of Bessel functions $J_n(x)$, the first term of Eq. (18) is written as

$$\begin{aligned} & e^{j(\omega_c t - \beta \cos(\omega_m t))} \\ &= e^{j\omega_c t} e^{j\beta \cos(\omega_m t - \pi)} \\ &= e^{j\omega_c t} \left[J_0(\beta) + 2 \sum_{n=1}^{\infty} (j)^n J_n(\beta) \cos n(\omega_m t - \pi) \right] \\ &= J_0(\beta) e^{j\omega_c t} + 2 \sum_{n=1}^{\infty} (j)^n J_n(\beta) [e^{j(\omega_c + n\omega_m)t - n\pi} \\ & \quad + e^{j(\omega_c - n\omega_m)t + n\pi}] \end{aligned} \quad (19)$$

Frequencies of components that satisfy $n\omega_m \geq \omega_c$ are negative. However, if the condition:

$$2J_n(\beta) \ll 1 \text{ for } n\omega_m \geq \omega_c \quad (20)$$

is satisfied, it can be considered that the first term of Eq. (18) contains only positive frequency components. Also, for the same reason, the second term of Eq. (18) contains only negative frequency components. Thus the first term of Eq. (18) is the analytic signal of Eq. (17) if

Eq. (20) is satisfied. Then, the instantaneous phase can be given as

$$\phi(t) = \omega_c t - \frac{\omega_d}{\omega_m} \cos(\omega_m t) \quad (21)$$

The instantaneous angular frequency is given by differentiating it with respect to time by

$$\omega_i(t) = \omega_c [1 + (\omega_d/\omega_c) \sin(\omega_m t)] \quad (22)$$

where ω_d/ω_c is the modulation depth.

An AM signal may be expressed as

$$x(t) = (1 + A \cos \omega_a t) \cos \omega_c t \quad (\omega_c > \omega_a) \quad (23)$$

where A is the amplitude-modulation index and ω_a is the modulation frequency. The analytic signal is then given by

$$\begin{aligned} x_a(t) &= e^{j\omega_c t} / 2 + (A/4) [e^{j(\omega_c + \omega_a)t} + e^{j(\omega_c - \omega_a)t}] \\ &= e^{j\omega_c t} [1 + (A/2)(e^{j\omega_a t} + e^{-j\omega_a t})] / 2 \\ &= e^{j\omega_c t} [1 + (A \cos \omega_a t)] / 2 \end{aligned} \quad (24)$$

resulting in an instantaneous angular frequency $\omega_i = \omega_c$.

3.4. Periodic Sinusoidal Burst Signals

So far we considered infinitely differentiable signals. However, it is interesting to see how the IF changes at and near the onset of a sinusoidal burst signal as shown in Fig. 1. The signal is periodic with period T and equal to zero during the first half of the period. The latter half of the period is a portion of a sinusoid starting with zero and with

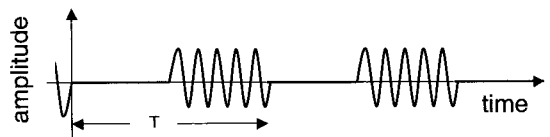


Fig. 1 A periodic sinusoidal burst signal.

an integer number of cycles (N_c). The periodic signal can be expressed using the Fourier series expansion by:

$$\begin{aligned}
 x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \\
 &= \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos(n\omega_0 t + \theta_n) \quad (25) \\
 \theta_n &= -\tan^{-1}(b_n/a_n)
 \end{aligned}$$

where $\omega_0 (= 2\pi/T)$ is the fundamental angular frequency of the periodic signal. By letting the angular frequency of the sine function be $\omega (= 2N_c\omega_0)$, the coefficients are given by

$$\begin{aligned}
 a_n &= \frac{2}{T} \int_{\pi/\omega_0}^{2\pi/\omega_0} \sin(2N_c\omega_0 t) \cos n\omega_0 t \, dt \\
 &= \frac{1}{2} \left[\frac{\cos(2N_c + n)2\pi - \cos(2N_c + n)\pi}{\pi(2N_c + n)} \right. \\
 &\quad \left. - \frac{\cos(2N_c - n)2\pi - \cos(2N_c - n)\pi}{\pi(2N_c - n)} \right] \quad (26) \\
 &= \begin{cases} \frac{1}{\pi(2N_c + n)} - \frac{1}{\pi(2N_c - n)} & (n: \text{odd}) \\ 0 & (n: \text{even}) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{2}{T} \int_{\pi/\omega_0}^{2\pi/\omega_0} \sin(2N_c\omega_0 t) \sin n\omega_0 t \, dt \\
 &= \frac{1}{2} \left[\frac{\sin(2N_c + n)2\pi - \sin(2N_c + n)\pi}{\pi(2N_c + n)} \right. \\
 &\quad \left. - \frac{\sin(2N_c - n)2\pi - \sin(2N_c - n)\pi}{\pi(2N_c - n)} \right] \quad (27) \\
 &= \begin{cases} 1/2 & (n: 2N_c) \\ 0 & (n: \text{else}) \end{cases}
 \end{aligned}$$

Now, we can apply Eq. (16) in order to evaluate the IF of the periodic sinusoidal burst signal. These expressions will be compared to results obtained by applying the discrete version of the analytic signal method described in the next section.

4. NUMERICAL EVALUATION OF IFS OF VARIOUS SIGNALS

4.1. Procedure for Calculating IFS —Discrete Method—

Steps shown in Sect. 2 are modified so that they can be applied to finite discrete signals $x(n)$.

- (1) Apply a fade-in and fade-out to the signal in order to avoid a possible discontinuity of the signal (this step is optional).
- (2) Apply a DFT to the signal.
- (3) Set the negative and DC components equal to zero.
- (4) Apply an IDFT to the above spectrum.
- (5) Obtain the IF by

$$f_i(n) = [\{\phi_i(n+1) - \phi_i(n-1)\}/2]/2\pi\Delta t \quad (28)$$

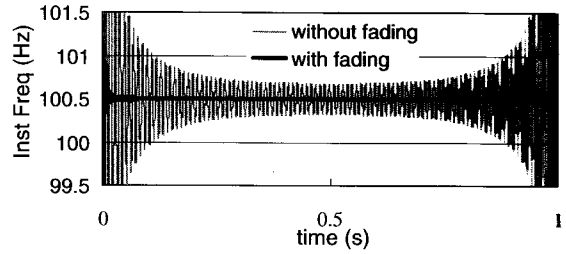


Fig. 2 IFs of a single sinusoid without and with fading.

The fade-in and fade-out may be achieved by applying the following window function to the signal:

$$w(n) = \begin{cases} \sin^2\left(\frac{\pi}{2} \frac{n}{N/16}\right) & (0 \leq n \leq N/16) \\ 1 & (N/16 < n < 15N/16) \\ \sin^2\left(\frac{\pi}{2} \frac{N-n}{N/16}\right) & (15N/16 \leq n \leq N-1) \end{cases} \quad (29)$$

where N is the sampling number used for the DFT analysis ($N/16$ was chosen as a rather arbitrary fade-in and fade-out time length).

4.2. Signals with a Single Frequency Component

The IF of a single frequency component signal with $a_1 = 1$, $f_1 = 100.5$ (Hz), $\theta_1 = \pi/3$, $f_s = 1/\Delta t = 8,192$ (Hz; sampling frequency), $N = 8,192$ (sampling number) is shown in Fig. 2 for cases with and without fade-in and fade-out. Since the time window length is 1 s, there are 100.5 cycles in the time window, which is the worst case of discontinuity. The thin and thick lines are for the cases without and with the fading. The result shows that the effect of the fading is significant. With the fading, the order of error around the center ($t = 0.5$ s) of the time window is approximately 0.001%, which is negligible for most practical applications. However, the accuracy near either end is not very good compared to near the center of the window. This does not improve much by varying the fade-in and fade-out time length from one-sixteenth of the window length.

The accuracy of the estimation also depends on the signal frequency. Estimation errors around the window center are ± 0.5 , ± 0.03 , $\pm 0.07\%$ for $f_1 = 10.5$, 30.5, and 1,000.5 Hz, respectively. It can be said that an existence of more than 30 cycles in the window gives enough accuracy (error within 0.05%). The error also increases as the signal frequency approaches the sampling frequency. Roughly speaking, the signal frequency should be kept below or equal to 1/10 of the sampling frequency.

It should be noted that, if the windowed sinusoid is continuous (i.e., if the number of cycles of the windowed signal is an integer), the error is practically zero (on the

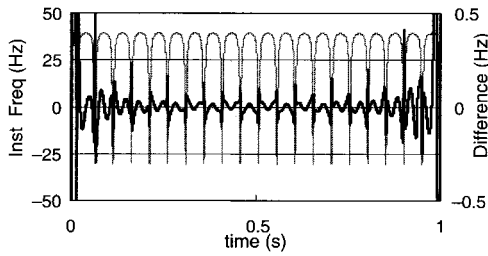


Fig. 3 The IF of a signal composed of two frequency components obtained by discrete method (thin line) and its difference obtained from the theoretical expression (Eq. (12)).

order of 10^{-10}) from the lowest possible frequency without the fade-in and fade-out. The fading in this case has an adverse (but tolerable) effect.

4.3. Signals with two Frequency Components

As Eq. (12) indicates, a signal with two frequency components can have IFs that are quite different than what we expect from a single frequency signal. In order to examine the validity of Eq. (12), the IFs of the signal with $a_1 = 1$, $f_1 = 30.5$, $\theta_1 = 0$, $a_2 = 3/4$, $f_2 = f_1 \times 5/3$ ($\cong 50.833$), and $\theta_2 = \pi/3$ were calculated by use of Eq. (12) and the numerical method shown in section 4.1. Results are shown in Fig. 3. The thin line shows the IF calculated by the numerical method and the thick line (with the right-hand-side vertical axis) shows its difference from the result obtained by use of Eq. (12). Errors are negligibly small except for those near both ends of the window, indicating the validity of Eq. (12). There are peaks and troughs, whose numbers correspond to the frequency difference of the two components of the signal.

The existence of negative IFs is mentioned in reference [4]. Figure 4 gives a clear explanation how the IF can take negative values. The dotted lines show two vectors corresponding to the two signal components and the added vector (indicated by an arrow) at one instance of time. At the next sampling time, the two vectors of the signal components rotate in the positive direction but with different phase increments as shown by solid lines. The added vector rotates in the negative direction, resulting in a negative IF. As a_1 gets closer to a_2 , the time interval with a negative frequency region gets narrower, and at the limit ($a_1 = a_2$) it disappears, giving a time-independent instantaneous frequency.

If we accept the definition of IF given in Sect. 2, we must accept its negative valued-ness as a consequence.

4.4. Frequency Modulated Signals

One of practical applications of IF is the analysis of vibratos of musical instruments [9]. In case of a violin, the

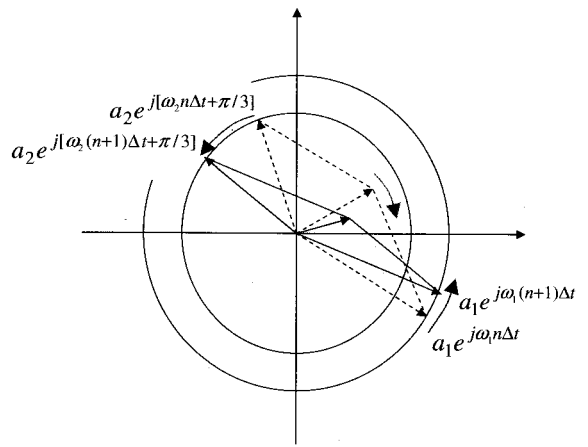


Fig. 4 Vector representations of a signal at two consecutive samplings $n\Delta t$ and $(n+1)\Delta t$.

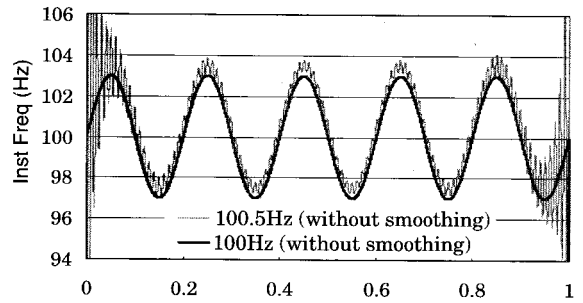


Fig. 5 IFs of frequency modulated sinusoids with two carrier frequencies.

harmonic components are frequency modulated with a modulation frequency of several Hz and a modulation depth of several percent. With this in mind, a frequency-modulated signal expressed by Eq. (17) was analyzed with $f_c = 100$ and 100.5 (Hz), $f_m = 5$ (Hz) and $f_d = 3$ (Hz). Results are shown by the thick and thin lines in Fig. 5, respectively. In case of $f_c = 100$, the error is within $\pm 0.00001\%$. However, for $f_c = 100.5$, the IF contains a small fluctuation of 100.5 Hz, which can be removed by applying a low-pass filter. Figure 6 shows the effect of smoothing using a rectangular low-pass filter with a cut-off frequency at 10 Hz. The thin and thick lines show IFs before and after the smoothing, respectively. The estimation error around the center of the window is within $\pm 0.01\%$. Figure 6 clearly shows that the IF of the frequency modulated signal is obtained with enough accuracy by the analytical signal method if the fading and smoothing are properly applied. It should be noted that the IF does not change even if a normal amount of amplitude modulation is introduced to the signal by combining Eqs. (17) and (23).

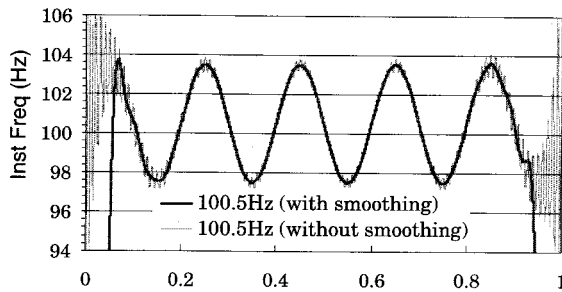


Fig. 6 The effect of smoothing on IFs of a frequency modulated sinusoid.

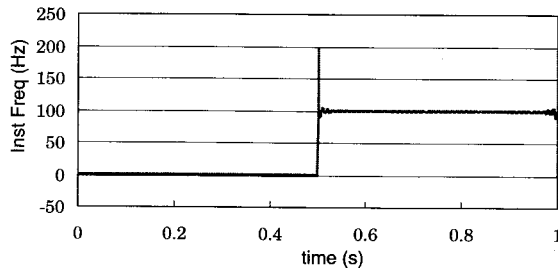


Fig. 7 The IF of a periodic sinusoidal burst signal ($T = 1$ (s) and $f = 100$ Hz) obtained by the discrete method.

4.5. Periodic Sinusoidal Burst Signals

The IF of a periodic sinusoidal burst signal shown in Fig. 1 with $T = 1$ (s) and $\omega = 2\pi \times 100$ (rad/s) was obtained by the discrete version of the analytical method with a seemingly large enough sampling number $N = 524,288$ ($= 2^{19}$) and sampling frequency $f_s = 524,288$ (Hz). The results for the whole period (0–1 s) are shown in Fig. 7 and detailed results around $t = 0.5$ s are shown by the thick line in Fig. 8. The IF is ideally zero (less than 10^{-10}) while the signal stays null ($0 < t < 0.5$ s). It jumps up to 99.4369... at $t = 0.5$ s and reaches its maximum ($= 198.0587...$) at the next sampling point ($= 0.5000019073...$) and then gradually approaches 100 with a small fluctuation. The IF behaves the same way near the end of the burst.

The IF of the same signal was also calculated by the Fourier series expansion method using Eqs. (23–24) and with $M = 20,000$ in Eq. (16). The result is shown by the thin line in Fig. 8 only in the time interval, $0.4995 \leq t \leq 0.5005$. The result obtained by the Fourier series expansion method shows fluctuations around the onset of the signal ($t = 0.5$ s) but is roughly the same with the result given by the discrete method. The similarity between the results obtained by the two completely different approaches, is a validation of Eq. (16). Figure 7 presents us with interesting findings. The IF is zero as long as the

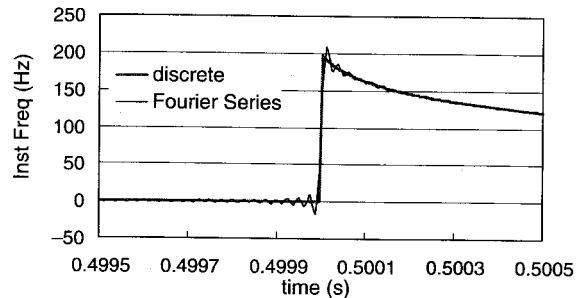


Fig. 8 IFs of a periodic burst sinusoid ($T = 1$ (s) and $f = 100$ Hz) obtained by the discrete method (thick line) and the Fourier series expansion method (thin line).

signal stays zero, and around the onset and the end of a sinusoidal burst, it takes a maximum value which is very close to twice that of the frequency of the sinusoid.

Appendix A provides a proof showing that the IF of a hemi-infinite sinusoid at the onset of the signal is exactly equal to twice that of the sinusoid frequency.

5. CONCLUSION

Properties of instantaneous frequencies of signals obtained by use of the analytical signal method were discussed. A general expression for the instantaneous frequency of a signal with multiple sinusoidal components was presented. In case of a signal with two sinusoids of the identical amplitude, the instantaneous frequency is equal to the mean of the two frequencies. If the two amplitudes are different, the instantaneous frequency varies periodically with the difference frequency of the two. A reason for the existence of negative instantaneous frequencies was given by a vector representation of sinusoidal components. For a sinusoidal burst signal (with frequency f_0), it was found that the instantaneous frequency stays zero during the period while the signal stays zero, and takes a value equal to $2f_0$ at the onset and the end of the burst, and f_0 during the remaining period of the burst. Generally, it was found that the accuracy of the present method was quite good. However, it was also shown that some precautions, such as a removal of the signal discontinuity and low-pass filtering, must be taken to improve the accuracy.

It seems that theoretical discussions of signal properties from the perspective of the instantaneous frequency (defined in the narrow sense) have not been fully investigated yet and there are more cases to which the instantaneous frequency can be successfully applied.

REFERENCES

- [1] L. R. Rabiner, "On the use of autocorrelation analysis for pitch detection," *IEEE Trans. Acoust. Speech Signal Process.*, ASSP-25, 24–33 (1977).

- [2] J. M. Baker, "A new time-domain analysis of human speech and other complex waveforms," *Ph.D. Thesis, CMU* (1975).
- [3] T. Abe, T. Kobayashi and S. Imai, "Robust pitch estimation with harmonics enhancement on noisy environments based on instantaneous frequency," *Proc. ICSLP '96*, Vol. 2, pp. 1277-1280 (1996).
- [4] L. Cohen, *Time-Frequency Analysis* (Prentice-Hall, Englewood Cliffs, NJ, 1995).
- [5] M. Mellody and G. H. Wakefield, "The time-frequency characteristics of violin vibrato: Modal distribution analysis and synthesis," *J. Acoust. Soc. Am.*, **107**, 598-611 (2000).
- [6] Y. Chisaki, H. Nakashima, S. Shiroshita, T. Usagawa and M. Ebata, "A pitch detection method based on continuous wavelet transform for harmonic signal," *Acoust. Sci. & Tech.*, **24**, 7-16 (2003).
- [7] L. B. Thomas, *Statistical Communication Theory* (Wiley, New York, 1969), pp. 650-652.
- [8] H. Suzuki, S. Morita and T. Shindo, "On the perception of phase distortion," *J. Audio. Eng. Soc.*, **28**, 570-574 (1980).
- [9] T. Wakayama, M. Nagashima, O. Yamazaki and H. Suzuki, "Comparison of violin vibratos among four virtuosos," *Proc. ISMA 2004*, 4-S2-2 (2004).
- [10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products* (Academic Press, New York, 1965), p. 407.

APPENDIX INSTANTANEOUS FREQUENCY OF A HEMI-INFINITE SINUSOID AT THE ONSET OF THE SIGNAL

The instantaneous frequency of a hemi-infinite sinusoid with frequency f_0 at $t = 0$ is discussed. The signal $x(t)$ is defined by

$$x(t) = u(t)z(t) \tag{A.1}$$

where

$$\begin{aligned} x_a(t) &= -\frac{j}{4} e^{j2\pi f_0 t} + \frac{1}{4} \left[\sin(2\pi f_0 t) + \frac{2j}{\pi} \left\{ \sin(2\pi f_0 t) \text{ci}(2\pi f_0 t) - \cos(2\pi f_0 t) \left(\text{si}(2\pi f_0 t) + \frac{\pi}{2} \right) \right\} \right] \\ &= \frac{1}{2} [\sin(2\pi f_0 t)] + \frac{j}{2} \left[-\frac{1}{2} \cos(2\pi f_0 t) + \frac{1}{\pi} \left\{ \sin(2\pi f_0 t) \text{ci}(2\pi f_0 t) - \cos(2\pi f_0 t) \left(\text{si}(2\pi f_0 t) + \frac{\pi}{2} \right) \right\} \right] \end{aligned} \tag{A.8}$$

where the integral formulae including cosine and sine integral functions, ci and si.

$$\begin{aligned} \int_0^\infty \frac{\sin(ax)}{\beta^2 - x^2} dx &= \frac{1}{\beta} \left[\sin(a\beta) \text{ci}(a\beta) - \cos(a\beta) \left(\text{si}(a\beta) + \frac{\pi}{2} \right) \right] \\ \int_0^\infty \frac{\cos(ax)}{b^2 - x^2} dx &= \frac{\pi}{2b} \sin(ab) \end{aligned} \tag{A.9}$$

were used [10]. The phase of this signal is then given by

$$\theta = \tan^{-1} y = \tan^{-1} \frac{\sin(s) \text{ci}(s) - \cos(s) (\text{si}(s) + \pi/2) - (\pi/2) \cos(s)}{\pi \sin(s)} \tag{A.10}$$

where $2\pi f_0 t$ was replaced by s for simplicity. The derivative of y with respect to s reduces to

$$\frac{dy}{ds} = \frac{\text{si}(s) + \pi}{\pi \sin^2(s)} \tag{A.11}$$

where

$$d \text{ci}(s)/ds = \cos(s)/s \quad \text{and} \quad d \text{si}(s)/ds = \sin(s)/s \tag{A.12}$$

$$u(t) = \frac{1}{2} [1 + \text{sgn}(t)] \tag{A.2}$$

with the sign function $\text{sgn}(t)$, and

$$z(t) = \sin 2\pi f_0 t \tag{A.3}$$

The Fourier transforms of Eqs. (A.2) and (A.3) are given, respectively, by

$$U(f) = \frac{1}{2} \left[\delta(f) + \frac{1}{j\pi f} \right] \tag{A.4}$$

and

$$Z(f) = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] \tag{A.5}$$

Then the Fourier transform of $x(t)$ is given by the convolution of $U(f)$ and $Z(f)$ as

$$\begin{aligned} X(f) &= \frac{1}{4j} \left[\delta(f - f_0) - \delta(f + f_0) \right. \\ &\quad \left. + \frac{1}{j\pi(f - f_0)} - \frac{1}{j\pi(f + f_0)} \right] \end{aligned} \tag{A.6}$$

By neglecting the negative frequency components and by inverse Fourier transforming, the analytic signal is obtained:

$$\begin{aligned} x_a(t) &= -\frac{j}{4} e^{j2\pi f_0 t} - \frac{1}{4\pi} \int_0^\infty \left[\frac{1}{(f - f_0)} - \frac{1}{(f + f_0)} \right] e^{j2\pi ft} df \\ &= -\frac{j}{4} e^{j2\pi f_0 t} + \frac{f_0}{2\pi} \int_0^\infty \frac{e^{j2\pi ft}}{(f_0^2 - f^2)} df \end{aligned} \tag{A.7}$$

This is rewritten as

were used. Then,

$$\frac{dy}{dt} = \frac{dy}{ds} \frac{ds}{dt} = \frac{2f_0\{\text{si}(s) + \pi\}}{\sin^2(s)} \quad (\text{A-13})$$

which gives

$$\frac{d\theta}{dt} = \frac{1}{1+y^2} \frac{dy}{dt} = \frac{\pi^2 2f_0\{\text{si}(s) + \pi\}}{\pi^2 \sin^2(s) + [\sin(s) \text{ci}(s) - \cos(s)\{\text{si}(s) + \pi/2\} - (\pi/2) \cos(s)]^2} \quad (\text{A-14})$$

Using

$$\lim_{s \rightarrow 0} \sin^2(s) = 0, \quad \lim_{s \rightarrow 0} \sin(s) \text{ci}(s) = 0 \quad \text{and} \quad \lim_{s \rightarrow 0} \text{si}(s) = -\pi/2 \quad (\text{A-15})$$

we obtain

$$\lim_{s \rightarrow 0} \frac{d\theta}{dt} = 4\pi f_0 \quad (\text{A-16})$$

which indicates that the IF of the hemi-infinite sinusoid with frequency f_0 is equal to $2f_0$ at $t = 0$.