

Model Analysis of a Hammer-String Interaction

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Abstract

A piano hammer-string interaction is analyzed by use of an equivalent electrical circuit. The hammer can be represented by an inductance and a capacitance, representing the hammer mass and the felt stiffness, respectively. The capacitance includes the nonlinear property of the felt. The driving point mobility of the string is calculated and then an equivalent electrical circuit of the string is obtained from the analogy. The interaction between the hammer and the string is then calculated numerically by solving a set of simultaneous equations step by step in the time domain. The solution enables to represent the motions of the hammer and the string by an animation. The piano hammer-string interaction is also analyzed taking into account a shank vibration. In the present model, the string is assumed perfectly rigid for simplicity. In the next step, however, the model will include a flexible string.

1. Introduction

There may be several purposes for analyzing a sound generating mechanism of musical instruments by use of physical or mathematical models. The most common one is probably to investigate the effect of various parameters on the tone quality. It is possible to change a single parameter independently from other parameters in the model analysis, which is actually impossible when a real instrument or a component of it is used.

Our major concern on the sound generation mechanism of a piano is to investigate the effect of touch on the tone quality. Our recent study on the effect of touch on the spectrum of a piano tones (G3, G4, and G5) shows that the touch may be able to affect the spectrum of at least tone G5[1]. One of the possible reasons for this is that the shank may be bent in different ways at the time of the attack depending on the difference of the touch. If the key is depressed hard at the beginning of the key stroke, the hammer is accelerated mostly at the onset of the motion and gains the speed in the early stage of its journey from the rest position till the interaction with the string. A large acceleration of the hammer may cause even a small amount of shank bending. On the other hand, if the key is pressed slowly at the beginning

and hard near the end of the key stroke, a degree of the shank bending at the time of the interaction with the string may be different from the previous case. At least, an observation of the hammer action shows that the shank is bent significantly after the hammer string interaction. Figure 1 shows one frame taken from a film of a hammer motion recorded using a high speed movie camera with 4,000 frames per second. It is impossible to see the bending of the shank from this picture, but it is clearly seen when it is observed during the motion.

The above experimental result as well as later studies of Askenfelt and Jansson [2] motivated us to analyze the hammer-string interaction by use of a physical model. In this series of studies, we intend to theoretically prove that the shank bending affects the spectrum of the string vibration and consequently the tone quality, which may lead to a conclusion that the touch has "audible" effects on the tone quality of the piano.

At first, the hammer-string interaction is analyzed assuming an ideal hammer, which is a simple mass-spring system. Secondly, the hammer-string interaction is analyzed including the shank bending with an assumption that the string is ideally rigid. The third step is a combination of the first two analyses, which is still under investigation.

A large portion of the study presented here was carried out by one of the authors during his stay at CBS Technology Center, Connecticut, USA [3-4]. Recalculations have been done lately with renewed programming and visualizations of results have been added.



Figure.1 One frame from a film of a key-action motion.

2. Hammer-string interaction

In the model analysis of the hammer-string interaction, the impedance analogy between the mechanical and electrical systems (force vs. voltage, velocity vs. current) is used [3]. The mechanical system is replaced by the equivalent electrical circuit and a set of simultaneous equations is obtained, which is solved in the time domain.

2.1. Hammer model

The hammer is modeled by a single-degree-of-freedom mass-spring system. The hammer mass is replaced by an inductor and the inverse of the spring constant, which depends on the deformation, is replaced by a capacitor. The relation between the hammer deformation and the acting force f is given by a fourth order polynomial:

$$f = K_1 x^2 + K_2 x^3 + K_3 x^4 \quad (1)$$

The hammer mass M_h and constants K_1 - K_3 are obtained from an experiment. The electrical model of the hammer is shown in Fig.2 which also includes that of the string.

2.2. String model

A string model in this study is obtained by simulating the driving point impedance of the string. The vibration of an ideal string, $y(x, t)$, excited by a sinusoidal force, $f(x, t)$, acting at point x_0 of the string and with an angular frequency ω can be described by:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y(x, t)}{\partial t^2} - \frac{f(x, t)}{T_0} \quad (2)$$

$$f(x, t) = F_0 \delta(x - x_0) e^{j\omega t}$$

where T_0 is the string tension and F_0 is the amplitude of the force and δ is Dirac's delta function. By replacing $y(x, t)$ with $Y(x)e^{j\omega t}$, we obtain:

$$T \frac{d^2 Y(x)}{dx^2} + \rho \omega^2 Y(x) = -F \delta(x - x_0) \quad (3)$$

The solution of Eq.(3) is expressed as a summation of individual eigenfunctions by use of the boundary conditions ($Y(0) = Y(L) = 0$, L : string length):

$$Y(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x / L) \quad (4)$$

with coefficients:

$$A_n = -2F_0 \sin(n\pi x_0 / L) / \{T_0 [\frac{\omega^2}{c^2} - (\frac{n\pi}{L})^2]\} \quad (5)$$

where $c = \sqrt{T/\rho}$ (ρ : linear density of the string). By comparing the driving point impedance $Y(x)/F_0$ with an

electrical system, it can be concluded that the driving point impedance is expressed by a circuit with a parallel connection of resonant branches with inductance M_n and capacitance C_n as shown in Fig.2.

$$M_n = \frac{T_0}{2c^2 \sin^2(n\pi x_0 / L)} \quad (6)$$

$$1/C_n = K_n = (\frac{n\pi}{L})^2 \frac{T_0}{2 \sin^2(n\pi x_0 / L)}$$

2.3. Combined model

Figure 2 is the whole hammer-string interaction model. While the hammer is moving freely, the switch is kept closed. The dc current going through the closed switch represents the constant speed of the hammer. At the time of contact, the switch is opened and interaction starts. The current going through each branch of the string part represents the velocity amplitude of the corresponding vibration mode. The electrical charge stored in each capacitor represents the displacement amplitude of each mode. Since we can calculate these currents, the motions of the hammer and the string can be known in detail.

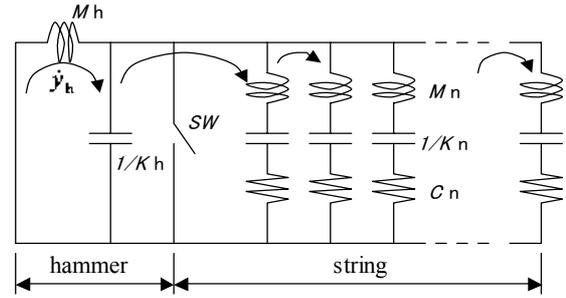


Figure.2 Electrical model of the hammer-string interaction.

2.4. Examples of results

One snapshot from the hammer-string interaction calculated with the parameters for note C4 is shown in Fig.3, where $M_h = 0.0078\text{kg}$, $K_1 = 0$, $K_2 = 0$, $K_3 = 3 \times 10^{14} (\text{N/m}^4)$, $L = 0.643\text{m}$, $x_0 = 0.114L$, $\rho = 0.0067\text{kg/m}$, $V_0 = 3\text{m/s}$ (initial hammer velocity) and $N = 30$ (max of n) [5]. The wave front is still traveling toward the right while the left part of the string is already in a vibratory motion.

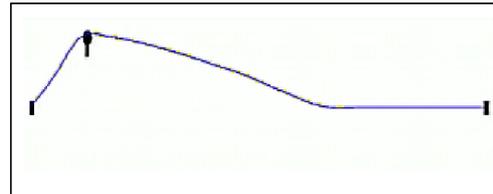


Figure.3 A snapshot from the hammer-string interaction calculated using the model shown in Fig.2.

Many other results can be obtained using this model, such as time histories of the hammer and the string motion and the force-time pattern applied by the hammer to the string and efficiency of the energy transmission from the hammer to the string. One of the interesting features of the modal expansion type modeling is that the effect of the inharmonicity is taken into account very easily even though it is in an approximate way. The stiffness can be increased gradually corresponding to the increase of the resonance frequencies of partials due to the inharmonicity. Figure 4 shows examples of wave propagations for a base string with and without the inharmonicity [3]. The right-side chart clearly shows the effect of dispersion. The waveform is gradually widened and deformed.

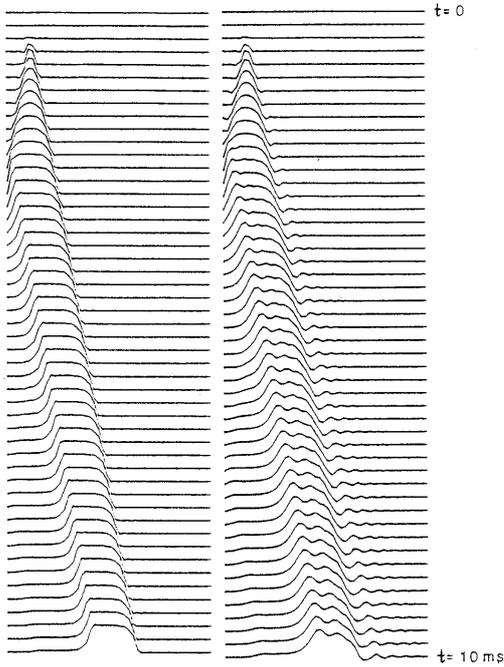


Figure.4 Comparison of wave propagations of strings with (right) and without (left) inharmonicity.

3. Shank motion analysis

3.1. Shank model

As discussed in section 1, the shank vibration is one of the possible reasons (or probably the "only" reason) that enable(s) the touch to affect the tone quality. With an intension to find out whether there is any effect of the initial bending of the shank on the force-time pattern applied to the string, the hammer was modeled including the shank as shown in Fig.5, where the string is assumed to be ideally rigid for simplicity [4]. The shank is divided into a number of elements as shown in Fig.5 (13 elements in this case). Each element is modeled by a pure mass and a mass-less beam as shown

in Fig.6. Symbols M , V , Y , θ indicate moment, shear force, displacement, and angle, respectively. Superscripts L and R indicate left- and right-hand side parameters, respectively.

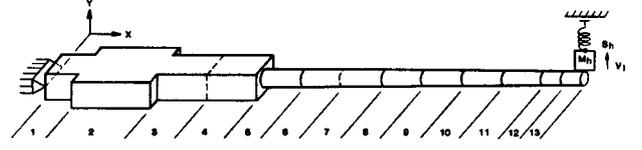


Figure.5 Model of the hammer-shank system.

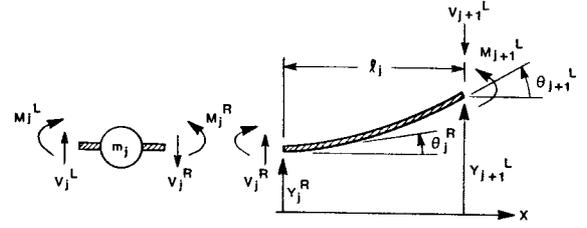


Figure.6 Representation of a single element by the mass and the mass-less beam.

Then the following equations hold for the j -th mass:

$$\begin{Bmatrix} V \\ M \\ \theta \\ Y \end{Bmatrix}_j^R = \begin{bmatrix} 1 & 0 & 0 & -m_j \frac{d^2}{dt^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V \\ M \\ \theta \\ Y \end{Bmatrix}_j^L \quad (7)$$

and for the j -th beam:

$$\begin{Bmatrix} V \\ M \\ \theta \\ Y \end{Bmatrix}_{j+1}^L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_j & 1 & 0 & 0 \\ \frac{l_j^2}{2B_j} & \frac{l_j}{B_j} & 1 & 0 \\ \frac{l_j^3}{6B_j} & \frac{l_j^2}{2B_j} & l_j & 0 \end{bmatrix} \begin{Bmatrix} V \\ M \\ \theta \\ Y \end{Bmatrix}_j^R \quad (8)$$

where B_j is the bending stiffness of the j -th element.

Substituting Eq.(7) into Eq.(8), we have:

$$\begin{Bmatrix} V \\ M \\ \theta \\ Y \end{Bmatrix}_{j+1}^L = \begin{bmatrix} 1 & 0 & 0 & -m_j \frac{d^2}{dt^2} \\ l_j & 1 & 0 & -m_j l_j \frac{d^2}{dt^2} \\ \frac{l_j^2}{2B_j} & \frac{l_j}{B_j} & 1 & -m_j \frac{l_j^2}{2B_j} \frac{d^2}{dt^2} \\ \frac{l_j^3}{6B_j} & \frac{l_j^2}{2B_j} & l_j & 1 - m_j \frac{l_j^3}{6B_j} \frac{d^2}{dt^2} \end{bmatrix} \begin{Bmatrix} V \\ M \\ \theta \\ Y \end{Bmatrix}_j^L \quad (9)$$

Eq.(9) can be rewritten as:

$$\begin{bmatrix} 1 & 0 & 0 & -m_j & -1 & 0 & 0 & 0 \\ l_j & 1 & 0 & -m_j l_j & 0 & -1 & 0 & 0 \\ \frac{l_j^2}{2B_j} & \frac{l_j}{B_j} & 1 & -m_j \frac{l_j^2}{2B_j} & 0 & 0 & -1 & 0 \\ \frac{l_j^3}{6B_j} & \frac{l_j^2}{2B_j} & l_j & -m_j \frac{l_j^3}{6B_j} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_j \\ M_j \\ \theta_j \\ \alpha_j \\ V_{j+1} \\ M_{j+1} \\ \theta_{j+1} \\ \alpha_{j+1} \end{bmatrix}^L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Y_{j+1} - Y_j \end{bmatrix}^L \quad (10)$$

where $\alpha_j = d^2 Y_j / dt^2$ was used. This equation holds for each element. The boundary conditions at the left and right ends of the shank are given, respectively, by:

$$\begin{aligned} \alpha_1^L &= 0 \quad (Y_1 \equiv 0) \\ M_1^L &= 0 \end{aligned} \quad (11)$$

and

$$\begin{aligned} V_{N+1}^L &= m_{N+1} \alpha_{N+1}^L + S_h Y_{N+1}^L \\ M_{N+1}^L &= 0 \end{aligned} \quad (12)$$

where m_{N+1} is the hammer mass and S_h is the hammer stiffness. Eqs.(10)-(12) make it possible to obtain acceleration vector $\{\alpha_j(0)\}$ ($j=1, \dots, N$) at the time of attack ($t=0$) if the initial displacement $\{Y_j(0)\}$ is given. Then, by using the following two equations, the displacement and velocity vectors $\{Y_j(\Delta t)\}$ and $\{v_j(\Delta t)\}$ at a discrete time $t = \Delta t$ can be obtained such as:

$$\begin{aligned} \{Y(\Delta t)\} &= \{Y(0)\} + \{v(0)\}\Delta t \\ \{v(\Delta t)\} &= \{v(0)\} + \{\alpha(0)\}\Delta t \end{aligned} \quad (13)$$

Eq.(13) indicates that if the initial displacement and the velocity distributions (vectors) along the shank are given, the whole process can be calculated using Eqs.(10)-(13).

3.2. Examples of results

Bending of the shank during the hammer head is in contact with the rigid string for a case without any initial shank deformation is shown in Fig.7, where $v_h = 3(\text{m/s})$, $m_{N+1} = 10(\text{g})$, $S_h = 1.8 \times 10^{10} Y_{N+1}^3 (\text{dyne/cm})$,

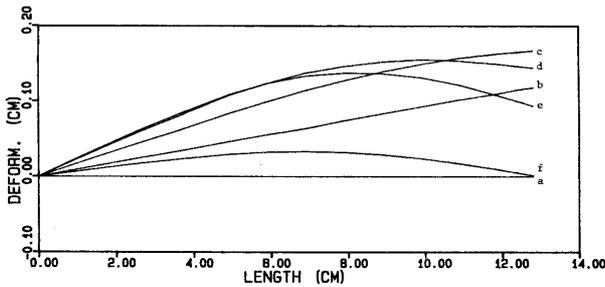


Figure.7 Deformation of the shank during the contact.

$E = 10^{11}(\text{dyne/cm}^2)$ (Young's modulus of the shank) and $\rho = 0.72(\text{g/cm}^3)$ (density of the shank) were used. Details of the shank dimensions are omitted. The maximum felt deformation occurs at $t = 0.71(\text{ms})$ (curve c) after the contact starts, and the interaction ends at $t = 1.4(\text{ms})$ (curve f). Force-time patterns with initial upward and downward shank deformations ($\pm 0.5\text{mm}$ at the mid point of the shank) are shown in Fig.8. In order to make the maximum forces equal, the hammer velocities are varied slightly. The figure shows that there is some effect of the initial bending on the force-time pattern.

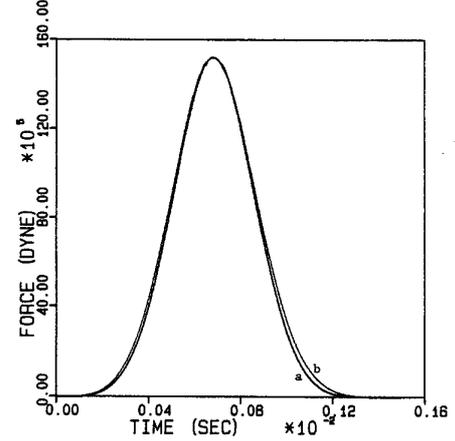


Figure.8 Force-time patterns with initial upward (a) and downward (b) shank deformations.

4. Future works

A hammer-string interaction with a flexible string including shank vibration is under investigation. An influence of the initial shank deformation on the exciting force will be made clear.

5. References

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