

PAPER

Curve fitting of frequency response functions by use of pole-zero models for the loss factor measurement

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Abstract: For the loss factor measurement, the half-power-width method and curve fitting method are used. In this paper, two curve fitting methods based on pole-zero models are proposed. The first method is to express the measured and model frequency response functions in decibels and the parameters of the model are changed so that the model response gets close enough to the measured response by the differential iteration method. The second method is to take the distance of the two responses on the complex plane and normalize it to the amplitude of the measured frequency response function. Then, the normalized distance is minimized by the differential iteration method. These two methods have the following features: (1) They can fit the resonance and anti-resonance characteristics simultaneously. (2) They can fit frequency response functions with large loss factors to which the half-power-width method are not applicable. (3) The minimum required numbers of measurement points within the half-power-width for these methods are roughly 1/30 of the number necessary for the half-power-width method.

Keywords: Loss factor, Curve fitting, Pole-zero model, Frequency response function

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1. INTRODUCTION

Due to the recent advancement of damping technologies, materials with large loss factors are available. The loss factors are obtained by measuring frequency response functions (abbreviated as FRF hereafter) of fixed-free beams or free-free beams [1] and applying the half-power-width method (abbreviated as HPWM hereafter). However, it can not be used if the neighboring vibration modes of the beam overlap with each other due to the large damping.

In recent years, the anti-resonance characteristics as well as the resonance characteristics of FRFs have been used to obtain loss factors [2]. The main reason is that the beam has impedance maxima at the anti-resonance, which makes the FRF measurement less vulnerable to the additional mass of the measurement system.

The second method used to obtain loss factors is the curve fitting method that is common in the field of modal analysis [3,4]. For a material with a large loss factor, a curve fitting with multi-degree-of-freedom models is necessary because of the modal overlap. However, the conventional method cannot fit the anti-resonance char-

acteristics because commonly used models are based on the resonance characteristics [5]. For these reasons, it was recommended to employ pole-zero models in order to curve fit FRFs [6]. Reference [6] shows that any FRF with negligible damping can be expressed by use of a parallel connection of resonance circuits or a series connection of anti-resonance circuits or a pole-zero model. If the damping is added to the circuits, however, only the pole-zero model can fit both the resonance and anti-resonance characteristics.

This paper reports two curve fitting methods which make it possible to fit both the resonance and anti-resonance characteristics of FRFs. The first one (named here as "decibel method") is to minimize the sum of the squared absolute of differences in decibels between the target (curve-fitted) FRF and the model FRF. The second method (named here as "normalization method") is to minimize the sum of the squared absolute of the difference between the target and the model FRFs in complex numbers. The difference is normalized to the power of the target FRF before taking summation. Numerical evaluations of the accuracy of the two methods are

discussed by simulation. Curve fitting examples to actually measured FRFs are also presented, and the results are compared with those obtained by HPWM.

2. VARIOUS DEFINITIONS OF THE LOSS FACTOR

Figure 1 shows an example of the experimental setup for the loss factor measurement. The driving point impedance (force/velocity) or the mobility (velocity/force) of the beam is measured and HPWM or the curve fitting method is applied to the measured FRFs. Since it is important to make it clear what is meant by the loss factor at least in this paper, it is briefly reviewed here.

2.1. Half-power-width Method (HPWM)

A loss factor of a beam obtained by HPWM is defined by

$$\eta_n = (f_{n2} - f_{n1})/f_{n0} \quad (1)$$

where f_{n0} are the resonance peak frequency of the n -th mode of the mobility, and f_{n1} and f_{n2} are the frequencies at which the response decreases by 3 dB from the resonance peak level.

2.2. Damping Ratio

The loss factor can be defined as two times of the damping ratio (ζ) as

$$\eta = 2\zeta = c/\sqrt{mk} \quad (2)$$

where m , k and c are mass, spring constant, and the viscous damping of the single-degree-of-freedom resonance system, respectively.

2.3. Modal Damping

If the n -th pole (resonance) of a driving point transfer function (velocity/force) is given by $s_n = -\alpha_n \pm j\beta_n$, the loss factor is defined by [3]

$$\eta_n = 2\alpha_n/\sqrt{\alpha_n^2 + \beta_n^2} \quad (3)$$

All the definitions give the same loss factor if the modal overlap is negligible. These definitions will be extended so that they can be applied to anti-resonance characteristics. For a anti-resonance system, Eq. (2) should be replaced by

$$\eta'_n = \sqrt{m_n k_n}/c_n \quad (4)$$

3. POLE-ZERO MODELS

The most common ‘‘mobility’’ model in the modal analysis is given by

$$Y(\omega) = \sum_{n=1}^N \frac{(j\omega/m_n)}{[-\omega^2 + j(\omega_n\eta_n)\omega + \omega_n^2]} + \frac{1}{j\omega m_0} + \frac{j\omega}{k_0} \quad (5)$$

where ω_n , η_n , and m_n are the angular resonance frequency, loss factor, and mass of the n -th mode, respectively, $1/k_0$ is the residual compliance and $-1/m_0$ is the inertia restraint. An electrical circuit equivalent to Eq. (5) based on the impedance analogy [7] is given by Fig. 2. The admittance of the circuit is given by Eq. (5). Each branch with m_n , $1/k_n$, and c_n , represents the n -th mode resonance indicating that the model is a resonance-based circuit. The anti-resonance characteristics result indirectly from the parallel connections of the adjacent resonant branches. This type of model does not have a freedom to independently determine both the resonance and the anti-resonance characteristics [6]. Actually, we can rewrite Eq. (5) in a pole-zero form with all parameters of zeros are expressed by the parameters of poles.

Another approach to the modeling of the driving point FRF is starting from the pole-zero model considering that all the poles and zeros are independent. There are four types of pole-zero models with alternating pole and zero frequencies. Those are $X_{A\pm}$ and $X_{B\pm}$ given by Eqs. (6) and (7).

$$X_{A\pm}(\omega) = \frac{H(\omega_1^2 - \omega^2 + j\omega_1\eta_1\omega) \cdots (\omega_{2N\pm 1}^2 - \omega^2 + j\omega_{2N\pm 1}\eta_{2N\pm 1}\omega)}{j\omega(\omega_2^2 - \omega^2 + j\omega_2\eta_2\omega) \cdots (\omega_{2N}^2 - \omega^2 + j\omega_{2N}\eta_{2N}\omega)} \quad (6)$$

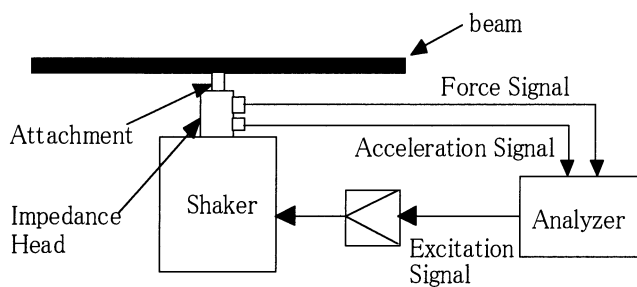


Fig. 1 Example of the loss factor measurement setup.

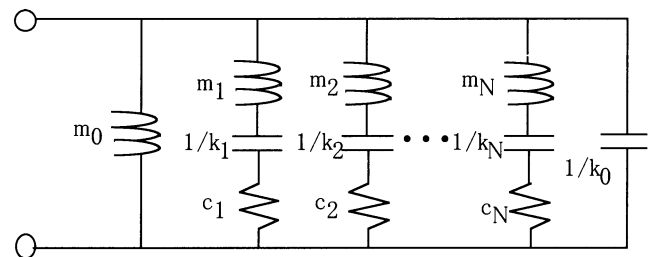


Fig. 2 Equivalent electrical circuit of the mathematical model defined by Eq. (5).

$$X_{B\pm}(\omega) = \frac{j\omega H(\omega_2^2 - \omega^2 + j\omega_2\eta_2\omega) \cdots (\omega_{2N}^2 - \omega^2 + j\omega_{2N}\eta_{2N}\omega)}{(\omega_1^2 - \omega^2 + j\omega_1\eta_1\omega) \cdots (\omega_{2N\pm 1}^2 - \omega^2 + j\omega_{2N\pm 1}\eta_{2N\pm 1}\omega)} \quad (7)$$

where H is a constant. $X_{A\pm}$ can model the driving point impedance of a fixed-free beam or the driving point mobility of a free-free beam since they have the first zero at a finite frequency, i.e., a pole at the zero (0) frequency. $X_{B\pm}$ can model the mobility of the fixed-free beam or the impedance of the free-free beam. The sign + or - represents whether the model has a pole or zero at the infinite frequency. X_{A+} is equivalent to Eq. (5) ("mobility" of the free-free beam with a pole at the infinite frequency). A significant difference between Eq. (5) and Eqs. (6) or (7) is that, if we start from Eq. (6) or (7), poles and zeros can be defined independently. Since we must fit a measured FRF (called as a target FRF hereafter) with the model FRF at both the resonance and anti-resonance, it is necessary to use a model defined by one of Eqs. (6) or (7).

In the electrical network theory, if the losses are completely zero, Eqs. (6) and (7) are called reactance functions. It has important properties such as [8]:

- 1) The poles and zeros repeat alternatively in the frequency domain.
- 2) They can be expressed by a parallel connection of L and C resonance circuits or by a series connection of L and C anti-resonance circuits.

From the second property, it is concluded that X_{A+} can be expressed in the form of Eq. (5) if the loss factors are zero. If the losses are added to the circuit, however, the two responses show a non-negligible difference with each other [4] even if the corresponding pole and zero frequencies are made the same. This is the limit of Eq. (5) as a mathematical model for a system with a finite damping.

4. CURVE FITTING METHODS

The general method of curve fitting is to minimize the error function λ defined below.

$$\lambda = \sum_{m=1}^M |A(\omega_m) - X(\omega_m)|^2 \quad (8)$$

where $A(\omega_m)$ and $X(\omega_m)$ are the target and model FRFs, respectively, and ω_m is the angular frequency of measurement. This type of error function cannot be used in order to curve fit both the resonance and anti-resonance characteristics because errors around the peaks (poles) of FRF contribute more to the error function and consequently the fitting around zeros may be worse percentage-wise. In order to solve this problem, two methods are tried here.

4.1. Decibel Method

The following error function is used in the decibel method.

$$\lambda_{dB} = \sum_{m=1}^M |A_{dB}(\omega_m) - X_{dB}(\omega_m)|^2 \quad (9)$$

where λ_{dB} is the error function, $A_{dB}(\omega_m)$ and $X_{dB}(\omega_m)$ are the target and model FRFs in decibels, respectively. By taking the differences in decibels, the model FRF should be fitted to the target FRF independently on the levels of the target FRF at individual frequencies.

The common method by which the error function is minimized in the frequency domain is the differential iteration method [3]. The initial values of the model parameters are roughly estimated, and the FRF of the model is linearized by means of Taylor expansion around the initial parameter set s such as

$$X_{ms\Delta} = X_{ms} + \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} \Delta\gamma_1 + \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} \Delta\gamma_2 + \cdots \quad (10)$$

where X_{ms} and $X_{ms\Delta}$ are the decibel values of the model FRF with the initial parameter set $s(\gamma_1, \gamma_2, \cdots)$ and with a parameter set $s(\gamma_1 + \Delta\gamma_1, \gamma_2 + \Delta\gamma_2, \cdots)$ both at frequency point m . $(\partial X / \partial \gamma_i)_{ms}$ is the derivative of $X_{dB}(\omega)$ with respect to parameter γ_i and then its parameters are substituted by the initial parameter set s .

For the simplicity of explanation, let's assume that the pole-zero model X_{B-} has just one pole and one zero.

$$X_{B-}(\omega) = \frac{j\omega H[(\omega_2^2 - \omega^2) + j\omega_2\eta_2\omega]}{[(\omega_1^2 - \omega^2) + j\omega_1\eta_1\omega]} \quad (11)$$

We can express X_{B-} in a decibel form (with the normalization factor equal to 1) such as

$$\begin{aligned} X_{dB}(\omega) = & 20 \log_{10} \omega + 20 \log_{10} H \\ & + 10 \log_{10} [(\omega_2^2 - \omega^2)^2 + (\omega_2\eta_2\omega)^2] \\ & - 10 \log_{10} [(\omega_1^2 - \omega^2)^2 + (\omega_1\eta_1\omega)^2] \end{aligned} \quad (12)$$

Then, derivatives with respect to each parameters are given by

$$\frac{X_{dB}(\omega)}{\partial \omega_2} = \frac{10}{\log_e 10} \frac{4(\omega_2^2 - \omega^2)\omega_2 + 2(\omega_2\omega\eta_2)\omega\eta_2}{(\omega_2^2 - \omega^2)^2 + (\omega_2\omega\eta_2)^2} \quad (13)$$

$$\frac{X_{dB}(\omega)}{\partial \eta_2} = \frac{10}{\log_e 10} \frac{+2(\omega_2\omega\eta_2)\omega\omega_2}{(\omega_2^2 - \omega^2)^2 + (\omega_2\omega\eta_2)^2} \quad (14)$$

$$\frac{X_{dB}(\omega)}{\partial H} = \frac{20}{\log_e 10} \left(\frac{1}{H}\right) \quad (15)$$

For a pole-zero model with more poles and zeros, derivatives corresponding to Eqs. (13) and (14) are given in the same form except that minus signs (-) must be added to the derivatives with respect to parameters of

poles.

Substituting Eq. (10) into Eq. (9) and letting the derivatives of λ_{dB} with respect to each parameters be zero, we obtain the following set of simultaneous equations,

$$[A]\{\Delta\gamma\} = \{B\} \quad (16)$$

where matrix $[A]$ and vector $\{B\}$ are given respectively by

$$[A] = \sum_{m=1}^M \begin{bmatrix} \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} & \dots \\ \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} & \dots \\ \left(\frac{\partial X}{\partial \gamma_3}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_3}\right)_{ms} & \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (17)$$

$$\{B\} = \sum_{m=1}^M \left\{ \begin{array}{l} (A_m - X_{ms}) \left(\frac{\partial X}{\partial \gamma_1}\right)_{ms} \\ (A_m - X_{ms}) \left(\frac{\partial X}{\partial \gamma_2}\right)_{ms} \\ (A_m - X_{ms}) \left(\frac{\partial X}{\partial \gamma_3}\right)_{ms} \\ \dots \end{array} \right\} \quad (18)$$

(Note: The summation symbol Σ applies to each components of the matrix and vector, and the variable ω is omitted.)

The initial parameters are now replaced with the new parameters:

$$\gamma'_i = \gamma_i + \mu \Delta\gamma_i \quad (19)$$

where μ is a convergence factor which controls the speed of convergence and the stability of iteration.

4.2. Normalization Method

The normalization method is a method which takes the target and model FRFs as complex numbers and the squared absolute of their difference is normalized to the power of the target FRF.

$$\lambda = \sum_{m=1}^M |A(\omega_m) - X(\omega_m)|^2 / |A(\omega_m)|^2 \quad (20)$$

By the above normalization, the contribution of each term becomes level-independent. The same differential iteration method can be applied to the normalization method with each terms of the matrix $[A]$ and vector $\{B\}$ in Eq. (16) given, respectively, by

$$a_{ij} = \sum_{m=1}^M \frac{\left(\frac{\partial X}{\partial \gamma_i}\right)_{mR} \left(\frac{\partial X}{\partial \gamma_j}\right)_{mR} + \left(\frac{\partial X}{\partial \gamma_i}\right)_{mI} \left(\frac{\partial X}{\partial \gamma_j}\right)_{mI}}{A_{mR}^2 + A_{mI}^2} \quad (21)$$

$$b_j = \sum_{m=1}^M \frac{(A_{mR} - X_{mR}) \left(\frac{\partial X}{\partial \gamma_j}\right)_{mR} + (A_{mI} - X_{mI}) \left(\frac{\partial X}{\partial \gamma_j}\right)_{mI}}{A_{mR}^2 + A_{mI}^2} \quad (22)$$

where R and I represent the real and imaginary parts of FRFs, respectively. The derivatives are also given in different forms compared to those given by Eqs. (13)–(15).

5. ACCURACY EVALUATION OF THE PROPOSED METHODS

If we use a measured FRF as the target FRF, we do not know exact values of the loss factors and, therefore, it is impossible to evaluate the accuracy of the algorithm. For this reason, the accuracy of the curve-fitting is evaluated here by synthesizing a target FRF from a set of known parameters.

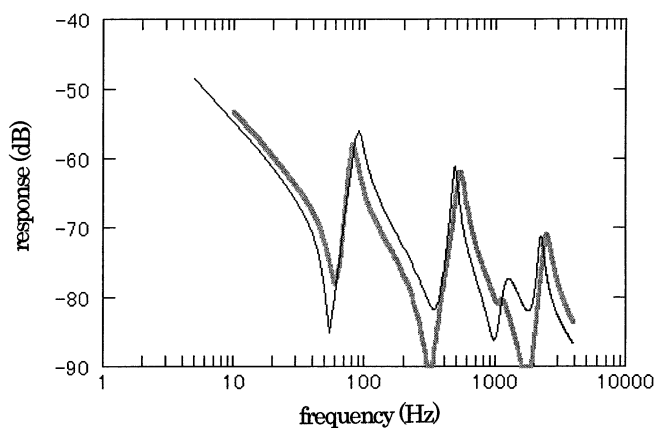
The second column of Table 1 shows true parameters of a pole-zero model with 4 poles and 4 zeros. Frequency ratios of the poles and zeros are those obtained theoretically and the loss factors are chosen arbitrarily from 0.1, 0.2, or 0.3. The third column shows the initial values used for the iteration. The frequencies are +10% or –10% off from the true values and the loss factors are all assumed to be equal to 0.15. The parameters of the pole-zero model after the convergence (40 iterations in this case) are given in the fourth and fifth columns of Table 1 for the decibel and the normalization methods, respectively. The errors of the decibel method is negligible to the order of 8 digits indicating that the error is negligible. The accuracy of the normalization method is not as good as the decibel method, but still the errors are small enough for a purpose of obtaining loss factors.

Figure 3 shows FRFs with the true parameters and with the initial parameters. The initial responses are significantly different. Even with this order of rough estimation, the algorithm shows a very good convergence. The final FRF is not shown since the target FRF and the model FRF after the iteration are too close to each other with the present graphical representation.

Another simulation was conducted for the purpose of finding the curve-fitting capability of the two algorithms with large loss factors. As the loss factor becomes larger, the level differences among the peaks and valleys become smaller and the curve fitting becomes more difficult. Figure 4 shows a dependence of the estimation error on the order of modes for three cases of loss factors $\eta = 0.7, 1.0,$ and 1.2 . The target FRFs were calculated with the same loss factor for all 8 poles and 8 zeros. The curve fitting

Table 1 True, initial, and estimated values of parameters.

	True parameters	Initial parameters	Estimated parameters (estimated/true)	
			Decibel method	Normalization method
Pole frequency	89.020	80.120	89.0200 (1.00000)	89.0269 (1.00008)
	481.05	529.17	481.050 (1.00000)	481.017 (0.99993)
	1187.9	1069.1	1187.90 (1.00000)	1186.54 (0.99886)
	2208.9	2429.8	2208.90 (1.00000)	2210.57 (1.00076)
Pole loss factor	0.2	0.15	0.20000 (1.00000)	0.20149 (1.00745)
	0.1	0.15	0.10000 (1.00000)	0.10087 (1.00870)
	0.3	0.15	0.30000 (1.00000)	0.30597 (1.01990)
	0.1	0.15	0.10000 (1.00000)	0.09848 (0.98480)
Zero frequency	55.980	61.570	55.9800 (1.00000)	55.9685 (0.99979)
	352.09	316.88	352.090 (1.00000)	351.801 (0.99918)
	992.84	1092.1	992.840 (1.00000)	991.189 (0.99833)
	1970.1	1773.0	1970.10 (1.00000)	1977.13 (1.00362)
Zero loss factor	0.1	0.15	0.10000 (1.00000)	0.09924 (0.99240)
	0.3	0.15	0.30000 (1.00000)	0.29786 (0.99287)
	0.2	0.15	0.20000 (1.00000)	0.19933 (0.99665)
	0.3	0.15	0.30000 (1.00000)	0.29882 (0.99607)
Constant H	1.000	—	1.00000 (1.00000)	1.00089 (1.00089)

**Fig. 3** FRFs with the true parameters (thin line) and with the initial parameters (thick line) given in Table 1.

using the decibel method was applied to the target FRF starting with 30% smaller initial loss factors. For the cases of the true loss factor at 0.7, 1.0, and 1.2, the curve fitting is possible with a negligible error (within 3%) up to 5th, 3rd, and 2nd mode, respectively. Figure 5 shows the target FRF (thin line) and the model FRF (thick line) for $\eta = 1.2$. It is interesting that even for a continuously increasing FRF (without any valleys) the curve fitting works well up to 2nd mode.

6. RESULTS OF CURVE FITTING TO MEASURED FRFS

It is interesting whether the pole-zero model defined by Eq. (6) or (7) can really fit to actually measured FRFs.

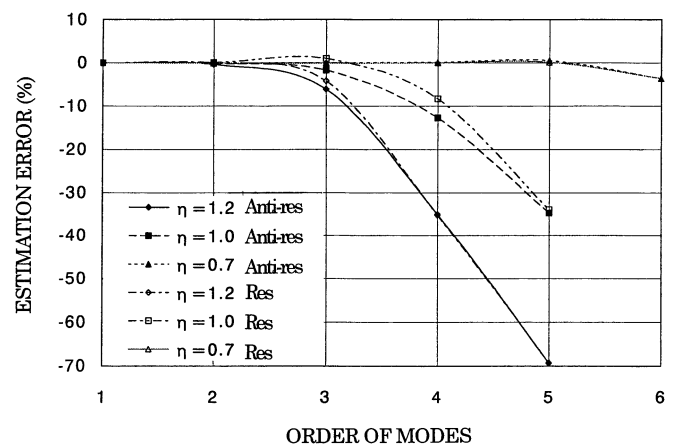
**Fig. 4** Dependence of the estimation error on the order of modes for three cases of loss factors ($\eta = 0.7$, 1.0, and 1.2).

Figure 6 shows an example of curve fitting to a measured FRF using the normalization method (thin line: target FRF, thick line: model FRF after curve fitting). Both numbers of poles and zeros of the model are set equal to 7. The highest pole and zero are kept outside of the frequency region of the curve fitting (frequencies of the 7th pole and zero are theoretically estimated values). The target FRF and the model FRF after the fitting are in good agreement with each other except the frequencies at which the target FRF seems to contain measurement errors. Since the true loss factors are not known, the results were compared to those obtained from the HPWM. The results are shown in Fig. 7. Differences of the “resonance” loss factors obtained by the two methods are within 10% and those of the “anti-

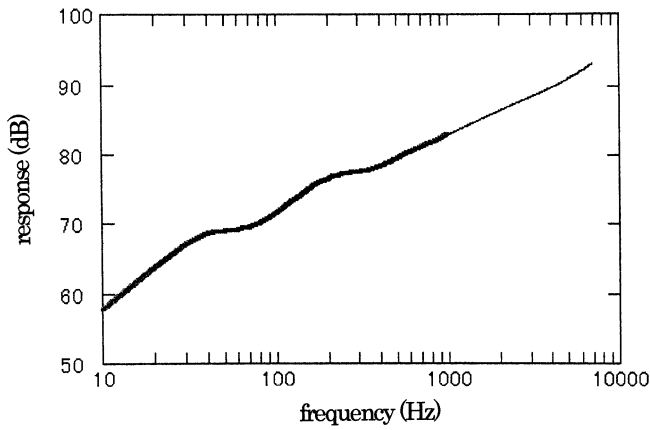


Fig. 5 Target (thin line) and the curve fitted (thick line) FRFs for $\eta = 1.2$.

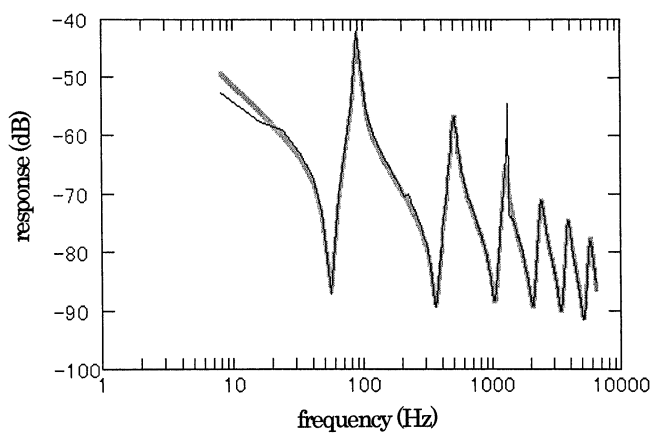


Fig. 6 Example of curve fitting to a measured FRF using the normalization method (thin line: target FRF, thick line: model FRF after fitting).

resonance” loss factors are slightly larger than 10%. It can be said that the agreement is reasonably good.

One of the main advantages of the curve fitting method is that the number of frequency points necessary for the curve fitting per mode can be much smaller than the number needed for HPWM. It has been reported that the minimum required “number of points within $\Delta f = (f_2 - f_1)$ per mode” (denoted as $NP_{-3\text{dB}}$, hereafter) is 10 if a loss factor with less than $\pm 2\%$ of error is required [2]. If the loss factor is 0.0002 at 100 Hz, the required frequency resolution of measurement is 0.002 ($= 0.0002 \times 100/10$), i.e., the time window length must be 500 s. This can be reduced significantly if a much smaller $NP_{-3\text{dB}}$ is required for the curve fitting method. Figure 8 shows how the percentage of the measurement error changes as the number of measurement points within Δf is varied. A steel beam was used, evaluating only the 4th anti-resonance (3,161.3 Hz). The “true” loss factor of this mode was assumed to be equal to 0.000231, which was obtained by curve fitting the measured data with $NP_{-3\text{dB}} = 15$. The

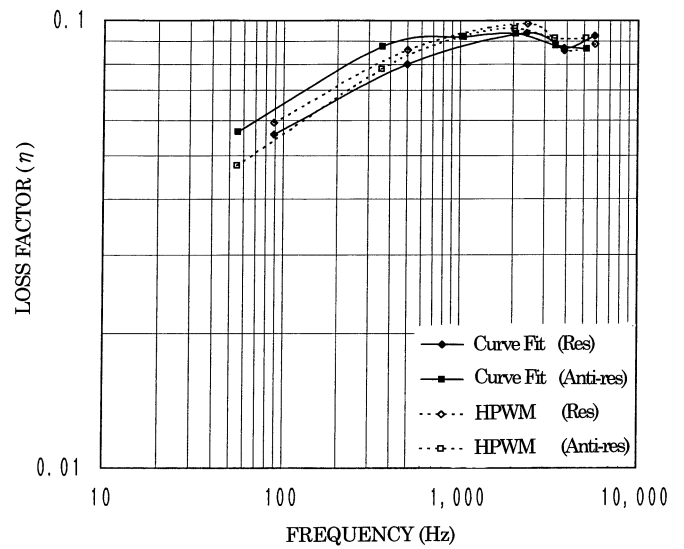


Fig. 7 Comparison of the “resonance” and “anti-resonance” loss factors obtained by the curve fitting method (normalization method) and half-power-width method (HPWM).

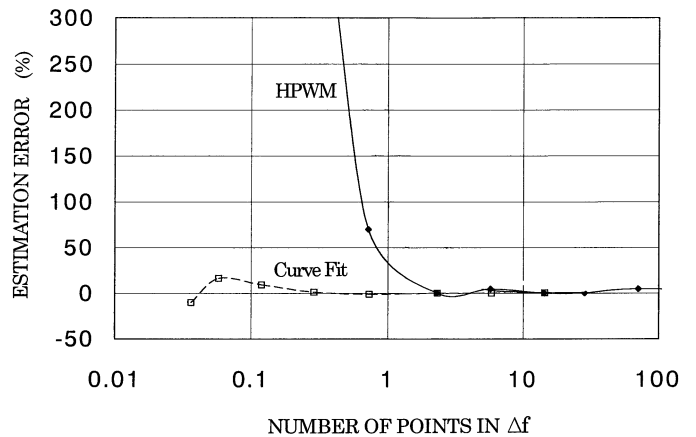


Fig. 8 Estimation error as a function of number of points within the half-power-width Δf (dotted line: curve fitting method, solid line: half-power-width method (HPWM)).

accuracy of this value was also confirmed by HPWM. In this example, it can be said that the minimum required $NP_{-3\text{dB}}$ for the curve fitting method is approximately 0.3 if a measurement error within $\pm 5\%$ is necessary, which is much smaller than that of $NP_{-3\text{dB}}$ required for HPWM. This contributes significantly to the reduction of the measurement time.

7. CONCLUSIONS

Pole-zero models were used to curve fit FRFs for the purpose of measuring loss factors both at resonance and anti-resonance of a beam. At first, the difference between the model conventionally used in the modal analysis and the proposed pole-zero model was discussed. The former does not have enough parameters to fit the resonance and

anti-resonance characteristics simultaneously. In the latter model, each resonance and anti-resonance can be expressed explicitly, and therefore, the curve fitting for both peaks and valleys are possible.

Two curve fitting methods were investigated. One is a method in which the error function is defined so that the squared difference in decibels between the target and the model FRFs are minimized. Another method is to minimize the difference between the target and the model FRFs in complex numbers, which is normalized to the power of the target FRF. Both methods were capable of fitting simulated FRFs very well. Two methods were also applied to measured FRFs. The fittings were very good as far as we could judge from the agreement of the two curves of the target and the model responses. The estimated loss factors were compared with the results obtained by HPWM. The agreement of the results were reasonably good.

The curve fitting method for the loss factor measurement is a must when materials with large loss factors are evaluated. The accuracy of the proposed methods should be investigated more in detail in order to avoid any miss-use in the loss factor measurement. A usefulness of the pole-

zero model for the curve fitting of FRFs in the general modal analysis should also be investigated in the future.

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