

Effect of inharmonicity of a piano on tuning

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INTRODUCTION

It is well known that a piano string has a property called inharmonicity. This is caused by the stiffness of the piano string. A stiff string has resonance frequencies that are not exactly integer multiples of the fundamental. The higher-order partials get sharper than the ideally harmonic case. This characteristic causes stretching of octaves in the base and treble regions of the piano scale. It also causes some difficulties of tuning, especially to a new tuner, since the beat rates found in handbooks are given for the ideal harmonic case and are not exactly true for a real piano with some degree of inharmonicity. A well-trained tuner may be able to tune any piano to its best condition by relying on his long experience by compromising differences between theory and reality. This chapter proposes one method that calculates beat rates by taking into account the effect of inharmonicity on tuning.

First, a measurement result of inharmonicity of a Steinway Model L grand piano is reported. The result shows that the inharmonicity proportionally increases with the key number in the F3-F4 octave. Second, the standard tuning procedure found in books is introduced and its problem, when applied to a real piano, is described. Lastly, a new method that takes into account the inharmonicity effect will be presented. (This study was conducted during 1981-1985 when the author was at CBS Technology Center, Stamford Connecticut, USA.)

INHARMONICITY AND ITS MEASUREMENT METHOD

It is well known that the stiffness of a piano string causes inharmonicity. When the string is deformed, there are two kinds of restoring forces, the tension of

the string and the force due to the stiffness. A simplified formula to calculate the resonance frequencies of partials is given by¹

$$f_n = F \cdot n(1 + Bn^2/2), \quad n = 1, 2, 3, \dots \quad (22.1)$$

and

$$B = (\pi^2 ESR^2 / TL^2) \quad (22.2)$$

where F is a constant, f_n is the resonance frequency of the n th mode, B is the inharmonicity index determined by the Young's modulus E , the cross-section S , the radius of gyration R , the tension T , and the string length L . Equation (22.1) can be rewritten as

$$f_n = f_1 \cdot n[1 + (Bn^2 - 1)/2], \quad n = 1, 2, 3, \dots \quad (22.3)$$

assuming that $B \ll 1$. It follows from eqn (22.1) that

$$F^2 = [(m/n)f_n^2 - \{(n/m)f_m\}^2] / [m^2 - n^2] \quad (22.4)$$

and

$$B = [(f_n/f_m)^2(m/n)^2 - 1] / [n^2 - (f_n/f_m)^2(m/n)^2 m^2] \quad (22.5)$$

Equation 22.5 gives a formula to obtain the inharmonicity from measured frequencies f_m and f_n . However, the accurate measurement of B is not an easy task for the following reasons:

- 1) The resonance frequencies of partials may not obey eqn (22.1) since the boundary condition at the bridge is not ideal.
- 2) The resonance frequencies vary slightly in time.
- 3) Extraneous resonances exist.
- 4) Durations (decay rates) of partials are different for different orders.

For low to mid frequency notes, it is possible to measure f_n for n up to 15 or 20. This is limited to $f_n \leq 10$ for high frequency notes. Since the frequencies of the lower order partials deviate less from the harmonic relationship than those of the high-order partials, the same frequency measurement error for lower orders causes a larger inharmonicity estimation error than for the higher orders. A careful measurement, with attention to the above mentioned problems, is necessary to obtain fairly accurate values of B for each note.

INHARMONICITY OF A STEINWAY GRAND PIANO

A measured characteristic of the inharmonicity index of a Steinway & Sons Model L grand piano is shown in Figure 22.1.² The keyboard is used as the horizontal axis showing the notes of the inharmonicity measurement. Figure 22.1 also shows the results for an upright piano measured by Fletcher.¹ Below key number 10, the grand piano has smaller B than the upright piano. Between key

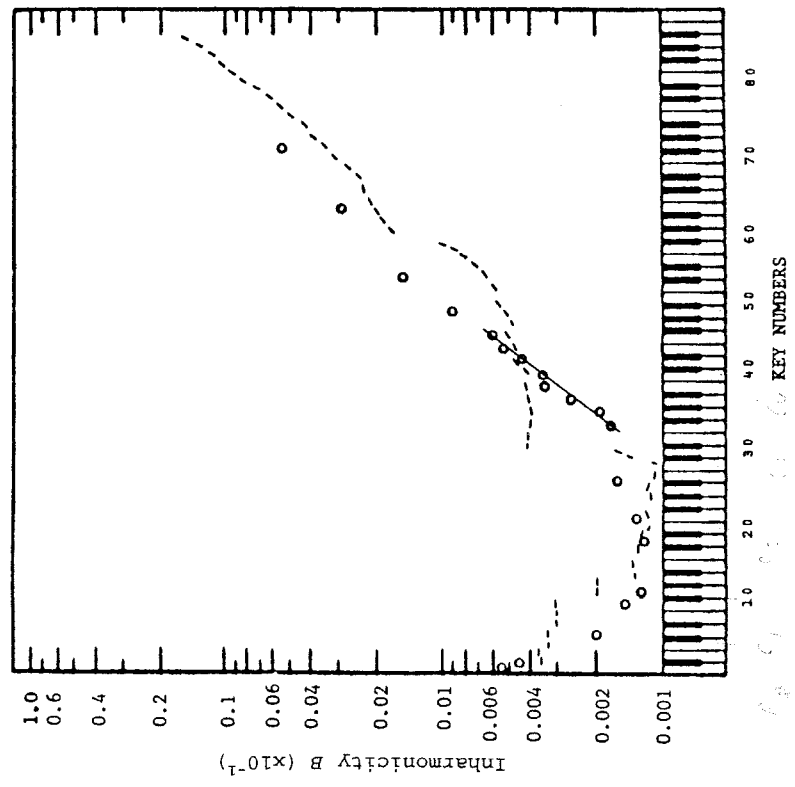


Fig. 22.1 Inharmonics of a Steinway Model L grand piano (circles) and an upright piano (dotted lines, after Fletcher).

numbers 10 and 30, both the grand and the upright pianos have approximately the same inharmonicity index. A transition from wound to unwound strings occurs between key number 26 (A#3) and 27 (B3) for the Model L grand piano and around key number 31 (D#3) for the upright piano, the latter showing a larger jump at the transition. Around key number 40, they have approximately the same inharmonicity and surprisingly the grand piano shows larger values for higher key numbers. The measured inharmonicity index of note C4 (of the grand piano) is approximately equal to the value calculated from eqn (22.2). For solid (unwound) strings, eqn (22.2) can be used to estimate the inharmonicity index instead of actual measurement of B by use of eqn (22.5).

The inharmonicity characteristic in the F3-F4 octave is approximated by a straight line. If integer numbers from 1 to 13 are assigned for each note from F3 to F4, the inharmonicity index B_n is given by

$$B_n = B_1 10^{((n-1)/12) \log(B_{13}/B_1)} \quad (22.6)$$

Values $B_1 = 0.00018$ and $B_{13} = 0.0006$ correspond to a straight line shown in

Figure 22.1. The approximated inharmonicity characteristic of eqn (22.6) will be used to discuss the effect of inharmonicity on tuning in the following sections.

STANDARD TUNING PROCEDURE

A piano can be tuned differently depending on the skill and/or the preference of a tuner (or a pianist). Some degree of difficulty in tuning a piano seems to be based on the inharmonicity of the piano strings. If there is no inharmonicity, the series of partials of a piano tone is ideally harmonic and tuning theory seems to work perfectly. Here, the 'theoretical tuning method' means a tuning method which utilizes only the beat rates obtained from an ideally harmonic series. The beat rates shown in textbooks are those theoretical beat rates (for example, see references 3 and 4). The steps of the tuning procedure given by Reblitz are: tuning fork \rightarrow C5 (beatless), C5 \rightarrow C4 (beatless), C4 \rightarrow F4 (3.0-), C4 \rightarrow G3 (4.5-), G3 \rightarrow D4 (3.5-), D4 \rightarrow A3 (5+), A3 \rightarrow E4 (3.5), E4 \rightarrow B3 (5.5), B3 \rightarrow F#3 (4+), F#3 \rightarrow C#4 (3+), C#4 \rightarrow G#3 (4.5+), G#3 \rightarrow D#4 (3.5+), D#4 \rightarrow A#3 (5.5-), and A#3 \leftrightarrow F3 (4+), F4 (4-). In the notation X \rightarrow Y (Z \pm), X and Y are reference and tuned notes, respectively, and Z is the beat per 5 seconds. The minus sign in the parenthesis indicates that the higher note between the tuning intervals has a partial frequency that is lower than the corresponding partial frequency of the lower note. The last step is a check step since both notes are tuned already. The same steps are shown in Figure 22.2. In the tuning procedure given by Reblitz, there are test and check processes during each step which are not shown here. Therefore, the actual tuning is much more laborious.

An issue the author would like to raise is that the beat rates shown above are obtained from theoretically calculated frequencies of partials and cannot be strictly applied to real pianos since any piano has some degree of inharmonicity as shown in the previous sections. If a tuner exactly follows the above steps (without compromise), he will find at the end that the beat rates between A#3 and F3 and F4 are far from what he expects. For example, if a constant inharmonicity $B = 0.0004$ is assumed in the F3-F4 octave (which is a fairly good assumption for some types of upright piano as seen in Figure 22.1) and the standard procedure described above is followed without checks, the beat per 5 seconds between F3 and A#3 is -12.7 and between A#3 and F4 is +6.6. These values are very different from the +4 and -4, respectively indicating that making compromises during the tuning steps is not an easy task.

The difficulty with aural tuning is that a tuner must tune one note to an already tuned note using a beat rate assigned to this specific tuning interval. If the beat rate were always zero, the tuning might be easier since he would not have to count the beat rate. For aural tuning, however, this is only the case when octave intervals are tuned. (Even in this case, a perfect zero-beat tuning

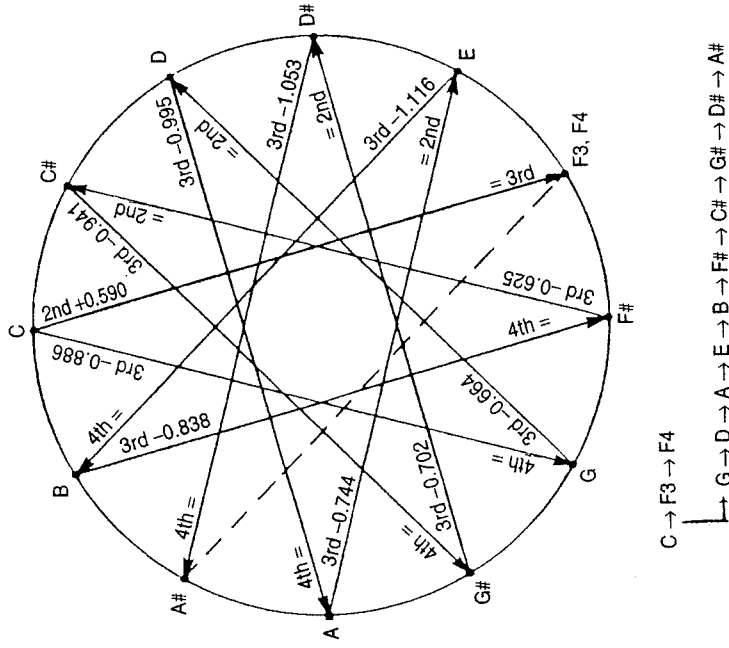


Fig. 22.2 Drawing to show the tuning procedure given by the theoretical tuning method.

is impossible because of the inharmonicity.) Another difficulty with aural tuning is that there is more than one beat when two notes of a tuning interval are played at the same time. Electronic tuning, on the other hand, is free from both of these problems. Thus if the piano tone does not have the inharmonicity property, electronic tuning may work perfectly well. The problem with electronic tuning is probably that octave stretching is necessary to properly tune an actual piano. Present-day electronic tuning devices are not designed to completely accommodate the stretched octave, even though they have an adjustment knob to slightly sharpen or flatten the pre-determined frequencies.

PROPOSED METHOD TO CALCULATE BEAT RATES FOR TUNING

A method of determining the frequencies of notes within the F3-F4 octave will be described which gradually shifts them to match the stretched F3-F4 octave. The beat rates calculated from these frequencies can be used for aural tuning as well as for electronic tuning.

First of all, note C4 is tuned to a tuning fork with a standard frequency of 523.252 Hz (or C5 already tuned to the tuning fork). Because of the inharmonicity of C4, the fundamental frequency of this note is slightly lower than $523.252/2 = 261.62$ Hz. In the present case, this is given as 261.485 Hz (assuming that the inharmonicity index of C4, $= B_8$, is equal to 0.00036 from eqn (22.6)), which is 0.93 cents lower than 261.626 Hz.

The next step is to determine the note frequencies in the range F3 to F4 with the reference frequency of C4 unchanged. Using eqn (22.3), the frequency ratios of F4/F3 and F5/F4 are given, respectively, by

$$X_1 = 2(1 + 1.5B_1) \quad (22.7)$$

and

$$X_2 = 2(1 + 1.5B_{13}) \quad (22.8)$$

If B_{13} is larger than B_1 (mostly this is true), the F4–F5 interval is stretched more than the F3–F4 interval. It seems appropriate to increase the frequency ratio of adjacent notes gradually to match the stretched intervals of F3–F4 and F4–F5. The condition that the frequency ratio of adjacent notes increases by a constant from F3 to F5 is enough to determine the frequencies from F3 to F4. The frequencies of notes F#4 and higher need not be determined at this stage, but this condition may ensure a smooth transition of the frequency ratio of adjacent notes from E4–F4 to F4–F#4. Figure 22.3 shows the frequency ratios of adjacent notes that satisfy the above condition. The initial frequency ratio of F#3/F3 is b . The next frequency ratio (G3/F#3) increases by a factor of a . Thus, it is equal to ab . The following ratio is a^2b , and so on. From these frequency ratios of adjacent notes, the frequency ratio of F4/F3 and F5/F4 are given by $a^{66}b^{12}$ and $a^{210}b^{12}$, respectively. Therefore, we have the relationship:

$$a^{144} = 1 + 1.5(B_{13} - B_1) \quad (22.9)$$

where B_{13} , $B_1 \ll 1$ is assumed. Since $(B_{13} - B_1)$ is very small, a is approximately given by

$$a = 1 + 0.010417(B_{13} - B_1) \quad (22.10)$$

Substituting eqn (22.10) into eqn (22.7) we have,

$$b^{12} = 2(1 - 0.6875B_{13} + 2.1875B_1) \quad (22.11)$$

Since B_{13} and B_1 are very small, b is approximately given by

$$b = 2^{1/12}(1 - 0.057292B_{13} + 0.182292B_1) \quad (22.12)$$

Using these two constants and starting from the reference frequency of C4, all other frequencies from F3 to F4 are determined. For example, the frequencies of B4 is given by f_c/a^6b , where f_c is the reference frequency of C4. The frequency of A#4 is given by $[f_c/a^6b]/a^5b$, and so on. The frequency of C#4 is given by $f_c \cdot a^7b$.

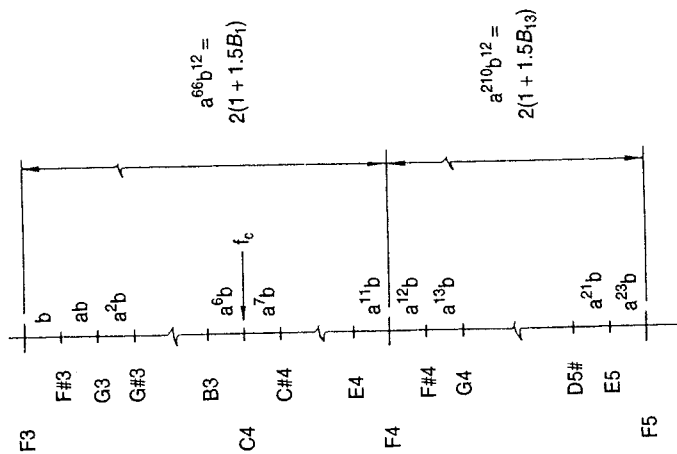


Fig. 22.3 Scale to show the frequency ratios of adjacent notes in the interval F3–F5.

Once the frequencies of the 13 notes from F3 to F4 are determined, beat rates which can be used for aural tuning are obtained. The result is shown in Table 22.1 for the case when the inharmonicity characteristic is given by the straight line in Figure 22.1. For example, the beat rate between F3 and C4 is 0.64/s. In Table 22.2, the beat rates of the ideally harmonic case are shown for comparison. The beat rates of the 5th intervals (F3–C4, F#3–C#4, ...) of the present case are slightly higher than the ideally harmonic case. The beat rates of 4th intervals (F3–A#4, F#3–B4, ...) in the present case are slightly lower than the ideally harmonic case. The 3rd and 6th intervals in the present case increase smoothly within like intervals. This property is almost always used as a check for proper tuning. Generally speaking it seems there is not much difference between Table 22.1 and Table 22.2. If the beat rates of Table 22.2 are used to tune a piano with the inharmonicity characteristic of the present case, however, unevenness in the beat rates results. Table 22.3 shows the result. The double boxed numbers are the beat rates used for the tuning and others are determined after the tuning. The 4th intervals F#3–B3 and C4–F4 that are not used for tuning have much smaller beat rates than the other 4th intervals. The 3rd intervals have uneven changes of beat rates within like intervals. This is the problem with theoretical tuning methods when they are applied to a real piano.

Table 22.3 Beat rates of intervals and cents deviation of each note (F3–F4) for $B_1 = 0.00018$, $B_8 = 0.00036$, and $B_{13} = 0.0006$ when the tuning steps C4–F3(–F4)–A#4–D#4–G#4–C#4–F#3 and C4–G3–D4–A4–E4–B4 are taken using the rates of the ideally harmonic case (double-boxed numbers)

											Cents dev		
											F	-0.78	
											E	-1.27	
											D#/Eb	-0.23	
											D	-1.13	
											C#/Db	-0.20	
											C	-1.23	
											B	-3.08	
											A#/Bb	-1.80	
											A	-2.42	
											G#/Ab	-1.18	
											G	-1.84	
											F#/Gb	-1.62	
F	.	.	-8.39	6.97	0.79	.	-0.59	.	7.96	.	.	0.0	-1.25
	.	.	-10.91	6.86	-0.27	.	-0.63	.	7.91	.	.	.	-1.62
	.	.	-10.90	6.84	0.89	.	-0.66	.	7.96	.	.	.	-1.84
	.	.	-13.43	7.66	0.94	.	-0.70	.	7.88	.	.	.	-1.18
	.	.	-12.30	9.29	1.00	.	-0.74	.	7.96	.	.	.	-2.42
	.	.	-13.09	8.54	1.05	.	-0.92	.	7.96	.	.	.	-1.80
	.	.	-13.91	10.38	1.12	.	-3.08	.	7.96	.	.	.	-3.08
	.	.	-15.95	8.57	0.228	.	-1.23	.	7.96	.	.	.	-1.23
	.	.	-19.22	8.43	.	.	-0.20	.	7.96	.	.	.	-0.20
	.	.	-19.23	.	.	.	-1.13	.	7.96	.	.	.	-1.13
	-0.23	.	7.96	.	.	.	-0.23
	-1.27	.	7.96	.	.	.	-1.27
	-0.78	.	7.96	.	.	.	-0.78

Table 22.4 Beat rates of intervals and cents deviation of each note (F3–F4) calculated from the suggested method for $B_1 = B_8 = B_{13} = 0.0004$

											Cents dev		
											F	-0.60	
											E	-0.69	
											D#/Eb	-0.78	
											D	-0.86	
											C#/Db	-0.95	
											C	-1.04	
											B	-1.12	
											A#/Bb	-1.21	
											A	-1.30	
											G#/Ab	-1.38	
											G	-1.47	
											F#/Gb	-1.56	
F	.	.	-11.59	5.56	-0.010	.	-0.93	.	5.55	.	.	0.0	-1.64
	.	.	-12.28	5.89	-0.010	.	-0.99	.	5.88	.	.	.	-1.56
	.	.	-13.01	6.24	-0.011	.	-1.05	.	6.23	.	.	.	-1.47
	.	.	-13.79	6.61	-0.012	.	-1.11	.	6.60	.	.	.	-1.38
	.	.	-14.61	7.00	-0.012	.	-1.17	.	6.60	.	.	.	-1.30
	.	.	-15.48	7.42	-0.013	.	-1.24	.	6.60	.	.	.	-1.21
	.	.	-16.40	7.86	-0.014	.	-1.12	.	6.60	.	.	.	-1.12
	.	.	-17.38	8.33	-0.015	.	-1.04	.	6.60	.	.	.	-1.04
	.	.	-18.41	8.83	.	.	-0.95	.	6.60	.	.	.	-0.95
	.	.	-19.51	.	.	.	-0.86	.	6.60	.	.	.	-0.86
	-0.78	.	6.60	.	.	.	-0.78
	-0.69	.	6.60	.	.	.	-0.69
	-0.60	.	6.60	.	.	.	-0.60

Table 22.1 Beat rates of intervals and cents deviation of each note (F3–F4) calculated from the suggested method for $B_1 = 0.00018$, $B_8 = 0.00036$ and $B_{13} = 0.0006$

											Cents dev												
											F	-0.61											
											E	.	-0.69										
											D# \flat /E \flat	.	.	-0.77									
											D	.	.	-16.73	-0.83								
											C# \flat /D \flat	.	.	-15.71	10.83	-0.89							
											C	.	.	-14.77	10.23	1.04	-0.94						
											B	.	.	-13.88	9.66	0.99	.	-0.98					
											A# \flat /B \flat	.	.	-13.06	9.13	0.94	.	-0.83	-1.02				
											A	.	.	-12.29	8.62	0.89	.	-0.78	.	-1.05			
											G# \flat /A \flat	.	.	-11.57	8.13	0.84	.	-0.74	.	9.18	-1.07		
											G	.	.	-10.90	7.68	0.80	.	-0.71	.	8.67	.	-1.08	
											F# \flat /G \flat	.	.	-10.27	7.24	0.75	.	-0.67	.	8.19	.	.	-1.08
F	.	.	-9.68	6.83	0.70	.	-0.64	.	7.73	.	.	.	0.0	.	.	-1.08							

Table 22.2 Beat rates of intervals of equal temperament (F3–F4) obtained from ideally harmonic partial frequencies

											F											
											E	.										
											D# \flat /E \flat	.	.									
											D	.	.	-15.85								
											C# \flat /D \flat	.	.	-14.96	11.00							
											C	.	.	-14.12	10.38	1.18						
											B	.	.	-13.33	9.80	1.12	.					
											A# \flat /B \flat	.	.	-12.58	9.25	1.05	.	-0.79				
											A	.	.	-11.87	8.73	1.00	.	-0.74	.			
											G# \flat /A \flat	.	.	-11.20	8.25	0.94	.	-0.70	.	9.42		
											G	.	.	-10.58	7.78	0.89	.	-0.66	.	8.89	.	
											F# \flat /G \flat	.	.	-9.98	7.34	0.84	.	-0.63	.	8.40	.	.
F	.	.	-9.42	6.93	0.79	.	-0.59	.	7.93	.	.	.	0	.	.							

