

## Control of inharmonicity by the addition of a point mass

Hideo Suzuki

*ONO SOKKI Co., Ltd.,*

*Shinjuku NS Bldg., 4-1 Nishishinjuku 2-Chome, Shinjuku-ku, Tokyo, 163 Japan*

*(Received 6 February 1989)*

A method for controlling the inharmonicity of a stiff string by adding a lumped mass (or masses) was originally suggested by F. Miller, Jr. In the present of paper, the same idea is investigated in a more precise manner. The effect of the string stiffness on its inharmonicity is discussed for different support conditions (the simple-simple, simple-fixed, and fixed-fixed conditions). The simple-simple support condition gives the largest inharmonicity (deviation from a true harmonic series) among them even though the differences are small for most cases. The theoretical and experimental results of the resonance frequencies of a string with the fixed-fixed condition show very good agreement with each other. For a representative middle C string, a mass of 0.2~0.4 grams, when attached 13~9 mm from one end of the string, reduces the inharmonicity significantly.

PACS number: 43. 72. Mn

### 1. INTRODUCTION

It is well known that frequencies of partials of a piano tone are not integer multiples of the fundamental resonance frequency. Both the theoretical and experimental aspects of the vibration of a uniform stiff string have been investigated.<sup>1-7)</sup> Lord Rayleigh<sup>1)</sup> and P. M. Morse<sup>2)</sup> contributed to the theoretical aspect of the vibration of a stiff string. R. S. Shankland and J. W. Coltman<sup>3)</sup> measured the resonance frequencies of a stiff string and compared them to the theoretical results obtained by Lord Rayleigh. O. H. Schuck and R. W. Young<sup>4)</sup> measured inharmonicity (deviation from a true harmonic) of piano tones for different key numbers and different types of pianos, and showed that a large (grand) piano is less inharmonic than a smaller piano. They also showed that the inharmonicity causes deviations in tuning (octave stretching). R. W. Young<sup>4)</sup> furthermore measured six pianos of various styles and makes, and raised a hypothesis that those pianos are "best" when the inharmonicity is least and also when it changes smoothly. F. Miller, Jr.<sup>6)</sup> discussed the possibility of controlling the inharmonicity by adding

a lumped mass near the end of the string. H. Fletcher<sup>7)</sup> covered both the theoretical and experimental aspects of inharmonicity.

The idea suggested by F. Miller, Jr.<sup>6)</sup> seems very attractive since it makes it possible to control the inharmonicity of a stiff string by adding a small mass near one end of the string. The addition of the lumped mass lowers the resonance frequency of each partial. For lower order partials, the effect of the mass is small since the mass is close to the nodes of the normal vibration of the string. As the order of the partial increases, the position of the anti-node approaches the position of the mass and, therefore, the mass works more effectively to lower the resonance frequency.

The present paper describes at first the theories used to calculate the resonance (eigen) frequencies and mode shapes of a uniform stiff string with and without the additional mass. Then, the resonance frequencies of a stiff string with different degrees of inharmonicity for the simple-simple, simple-fixed, and fixed-fixed conditions are numerically calculated without approximation. Finally, the theoretical and experimental results of resonance

frequencies of a stiff string with an additional mass of different sizes and at different position on the string are compared using an experimental setup which gives a fixed-fixed support condition.

**LIST OF SYMBOLS**

- $A_n$ : coefficient of mode function  $Y_n(x)$
- $B$ : parameter defining the inharmonicity of a string (Eq. (7))
- $C$ : mass ratio  $M_a/l\mu$
- $D$ : diameter of a string ( $= 2R$ )
- $E$ : Young's modulus of a string
- $K_1-K_4$ : coefficients defined by Eq. (9)
- $L$ : operator defined by Eq. (3)
- $M_a$ : additional lumped mass
- $R$ : radius of a string
- $S$ : area of the cross-section of a string ( $= \pi R^2$ )
- $T$ : tension of a string
- $Y(x)$ : displacement function of a string
- $Y_n(x)$ :  $n$ -th mode shape of a string
- $f_n$ :  $n$ -th resonance frequency of a string
- $f_0$ : fundamental resonance frequency of an ideal string
- $j$ : unit of imaginary number  $\sqrt{-1}$
- $k, k_1, k_2$ : wave numbers
- $l$ : string length
- $\Delta l$ : distance of the position of the lumped mass from the near end
- $m, n$ : mode numbers
- $t$ : independent time parameter
- $x$ : coordinate in the length direction of a string
- $x_0$ : position of the additional mass
- $y$ : time dependent displacement function
- $\delta(x)$ : delta function
- $\mu$ : linear density
- $\pi$ : circular constant
- $\omega$ : angular frequency
- $\omega_n$ :  $n$ -th mode angular frequency

**2. MATHEMATICAL DISCUSSION**

**2.1 Resonance Frequencies and Mode Shapes**

The basic equation of motion of a homogeneous stiff string is<sup>7)</sup>

$$ESR^2 \frac{\partial^4 y(x, t)}{\partial x^4} + \mu \frac{\partial^3 y(x, t)}{\partial t^2} - T \frac{\partial^2 y(x, t)}{\partial x^2} = 0 \quad (1)$$

For a steady state motion, the time dependent term  $e^{j\omega t}$  can be avoided using  $y = Y(x) e^{j\omega t}$ , such as

$$ESR^2 \frac{d^4 Y(x)}{dx^4} - T \frac{d^2 Y(x)}{dx^2} - \mu \omega^2 Y(x) = 0 \quad (2)$$

Define the operator  $L$  such that

$$L = \left( ESR^2 \frac{d^4}{dx^4} - T \frac{d^2}{dx^2} \right), \quad (3)$$

then

$$L\{Y(x)\} = \mu \omega^2 Y(x). \quad (4)$$

Assuming a solution of the form  $e^{2\pi kx}$  and substituting this into Eq. (2), we have

$$k^4 - (1/4Bl^2)k^2 - (f^2/16Bl^4f_0^2) = 0. \quad (5)$$

Solving Eq. (5), we have four solutions:

$$k = \pm k_1$$

and

$$k = \pm ik_2,$$

where

$$k_1 = \left[ \frac{1}{8Bl^2} \left\{ \left( 1 + \frac{4Bf^2}{f_0^2} \right)^{1/2} + 1 \right\} \right]^{1/2} \quad (6a)$$

and

$$k_2 = \left[ \frac{1}{8Bl^2} \left\{ \left( 1 + \frac{4Bf^2}{f_0^2} \right)^{1/2} - 1 \right\} \right]^{1/2} \quad (6b)$$

The notations

$$B = (\pi^2 ESR^2/Tl) \quad (7)$$

and

$$f_0 = (T/\mu)^{1/2}/2l \quad (8)$$

were also used.

The general solution of Eq. (2) is then given by

$$Y(x) = K_1 \cosh 2\pi k_1 x + K_2 \sinh 2\pi k_1 x + K_3 \cos 2\pi k_2 x + K_4 \sin 2\pi k_2 x. \quad (9)$$

The boundary conditions are expressed as follows.

For the fixed boundary condition,

$$\left. \begin{aligned} Y(x) &= 0 \\ \frac{dY(x)}{dx} &= 0 \end{aligned} \right\} \text{ at } x = -\frac{l}{2} \text{ or } \frac{l}{2} \quad (10)$$

For the simply supported boundary condition,

$$\left. \begin{aligned} Y(x) &= 0 \\ \frac{d^2 Y(x)}{dx^2} &= 0 \end{aligned} \right\} \text{ at } x = -\frac{l}{2} \text{ or } \frac{l}{2} \quad (11)$$

Four equations given by Eq. (10) and/or Eq. (11) determine the resonance frequencies  $f_n$  and the corresponding mode shapes  $Y_n(x)$ . These results will be used to obtain the resonance frequencies and mode shapes of a stiff string with an additional lumped mass.

H. SUZUKI: CONTROL OF INHARMONICITY

2.2 Eigenfrequencies and Eigenfunctions of a String with an Additional Lumped Mass

Assuming that the resonance frequencies and mode shapes of a uniform string are known, the deflection of the string with an additional lumped mass is expressed by:

$$Y(x) = \sum_{n=1}^{\infty} A_n Y_n(x), \tag{12}$$

where

$$L\{Y_m(x)\} = \mu\omega_m^2 Y_m(x) \tag{13}$$

and

$$\int_{-l/2}^{l/2} Y_m(x) Y_n(x) dx = l\delta_{mn}. \tag{14}$$

Equation (14) is obtained using the orthogonal property of the eigenfunctions  $Y_m(x)$  and letting the norm of each function equal to  $l$ .

The equation of motion with the additional mass is

$$L\{Y(x)\} - \mu\omega^2 Y(x) = -j\omega M_a \delta(x-x_0) \cdot j\omega Y(x). \tag{15}$$

Substituting Eq. (12) into Eq. (15), we have

$$L\left\{\sum_{n=1}^{\infty} A_n Y_n(x)\right\} - \mu\omega^2 \sum_{n=1}^{\infty} A_n Y_n(x) = M_a \omega^2 \delta(x-x_0) \sum_{n=1}^{\infty} A_n Y_n(x). \tag{16}$$

Multiplying by  $Y_m(x)$  and integrating over  $[-l/2, l/2]$ ,

$$L\left\{\sum_{n=1}^{\infty} A_n \int_{-l/2}^{l/2} Y_n(x) Y_m(x) dx\right\} - \mu\omega^2 \sum_{n=1}^{\infty} A_n \int_{-l/2}^{l/2} Y_n(x) Y_m(x) dx = M_a \omega^2 \sum_{n=1}^{\infty} A_n \int_{-l/2}^{l/2} Y_n(x) Y_m(x) \delta(x-x_0) dx \tag{17}$$

Using Eqs. (13) and (14)

$$l\mu(\omega_m^2 - \omega^2)A_m = M_a \omega^2 \sum_{n=1}^{\infty} A_n Y_n(x_0) Y_m(x_0). \tag{18}$$

Therefore,

$$\omega^2 \left\{ A_m + \left( \frac{M_a}{l\mu} \right) \sum_{n=1}^{\infty} A_n Y_n(x_0) Y_m(x_0) \right\} = \omega_m^2 A_m, \quad m=1, 2, \dots \tag{19a}$$

or in a matrix form:

$$\begin{bmatrix} \omega^2\{1 + CY_1^2(x_0)\} - \omega_1^2 & \omega^2 CY_1(x_0)Y_2(x_0) & \dots & \dots \\ \omega^2 CY_2(x_0)Y_1(x_0) & \omega^2\{1 + CY_2^2(x_0)\} - \omega_2^2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \dots \\ \dots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ \dots \end{Bmatrix}, \tag{19b}$$

where  $C = M_a/l\mu$ .

Equation (19) indicates that the resonance frequencies and mode shapes with the additional mass are determined by the resonance frequencies and mode shapes of the homogeneous stiff string, the ratio of the additional mass to the string mass, and the position of the additional mass on the string.

3. RESULTS AND DISCUSSION

3.1 Resonance Frequencies and Modal Shapes of a Homogeneous String

Using Eq. (9) and the boundary conditions expressed by Eqs. (10) and/or (11), the resonance frequency of a stiff string normalized to the fundamental resonance frequency of an ideal string (Eq. (8)) were calculated for three representative values of  $B$ . The results are shown in Table 1 ~ Table 3. For convenience, the fixed and simply supported bounda-

ry conditions are expressed simply by "F" and "S," respectively. It is reasonable that the F-F support gives the highest resonance frequencies among three different kinds of support. For  $B=0.001$  (which may be a typical inharmonicity index of note G#5 (key #60) of an upright piano, according to the reference,<sup>7)</sup> the deviation of  $f_{20}$  from  $20f_1$  is approximately 290 cents (18.25%) for any type of support condition.

By looking at the tables carefully, the following can be concluded:

1. The resonance frequency of a stiff string for the S-F support is approximately the geometrical mean of the frequencies for the S-S and F-F supports.
2. For  $B=0.0001$  and  $0.001$ , the ratios between adjacent frequencies (frequency spacings) are

**Table 1** Resonance frequencies of a stiff string with  $B=0.0001$  normalized to  $f_0=(T/\mu)^{1/2}/2l$ .

Mode number	S-S	Boundary conditions S-F	F-F
1	1.00005	1.0032	1.0065
2	2.0004	2.0068	2.0132
3	3.0014	3.0109	3.0206
4	4.0032	4.0160	4.0289
5	5.0063	5.0222	5.0383
6	6.0108	6.0300	6.0493
7	7.0171	7.0396	7.0621
8	8.0256	8.0512	8.0770
9	9.0364	9.0653	9.0943
10	10.0499	10.0820	10.1144
11	11.0663	11.1018	11.1374
12	12.0861	12.1248	12.1637
13	13.1094	13.1514	13.1936
14	14.1365	14.1818	14.2274
15	15.1678	15.2164	15.2653
16	16.2035	16.2554	16.3077
17	17.2439	17.2992	17.3548
18	18.2893	18.3479	18.4070
19	19.3399	19.4020	19.4644
20	20.3961	20.4616	20.5274

**Table 2** Resonance frequencies of a stiff string with  $B=0.001$  normalized to  $f_0=(T/\mu)^{1/2}/2l$ .

Mode number	S-S	Boundary conditions S-F	F-F
1	1.0005	1.0107	1.0211
2	2.0040	2.0224	2.0452
3	3.0135	3.0442	3.0775
4	4.0319	4.0730	4.1149
5	5.0621	5.1138	5.1665
6	6.1070	6.1694	6.2331
7	7.1695	7.2428	7.3176
8	8.2520	8.3366	8.4227
9	9.3574	9.4534	9.5511
10	10.4881	10.5957	10.7053
11	11.6465	11.7661	11.8879
12	12.8350	12.9668	13.1010
13	14.0556	14.2000	14.3469
14	15.3106	15.4678	15.6276
15	16.6020	16.7722	16.9452
16	17.9314	18.1150	18.3015
17	19.3008	19.4980	19.6982
18	20.7117	20.9227	21.1369
19	22.1658	22.3909	22.6192
20	23.6643	23.9037	24.1465

**Table 3** Resonance frequencies of a stiff string with  $B=0.01$  normalized to  $f_0=(T/\mu)^{1/2}/2l$ .

Mode number	S-S	Boundary conditions S-F	F-F
1	1.0050	1.0381	1.0734
2	2.0396	2.1069	2.1786
3	3.1321	3.2358	3.3458
4	4.3081	4.4507	4.6014
5	5.5902	5.7744	5.9681
6	6.9971	7.2254	7.4644
7	8.5446	8.8191	9.1053
8	10.2450	10.5675	10.9023
9	12.1083	12.4801	12.8643
10	14.1421	14.5643	14.9996
11	16.3527	16.8258	17.3124
12	18.7446	19.2693	19.8075
13	21.3216	21.8981	22.4883
14	24.0865	24.7151	25.3574
15	27.0416	27.7224	28.4169
16	30.1887	30.9281	31.6685
17	33.5293	34.3145	35.1134
18	37.0643	37.9017	38.7528
19	40.7947	41.6843	42.5876
20	44.7214	45.6631	46.6183

approximately the same for the three different kinds of support.

- For  $B=0.01$ , it is clearly seen that the frequency spacing is widest for the S-S support and narrowest for the F-F support even though the differences are very small.

What conclusion (1)~(3) indicate is very interesting. Contrary to usual reasoning, the S-S support condition gives the same or a larger inharmonicity than the other two support conditions do. An observation of the modal shapes may give a hint to this reason.

The modal shapes of a stiff string with the S-S support are exactly the same as the modal shapes of an ideal string.<sup>7)</sup> The fixed boundary condition at one end or at both ends causes the modal shapes to deviate from those of the ideal string. As an example, the 4th mode of a stiff string with the F-F support and  $B=0.001$  is shown by the curve with smaller slopes at both ends in Fig. 1. The other curve in the figure is the modal shape of the ideal string. It can be seen that the fixed boundary condition reduces the effective length of the string. This makes the resonance frequencies of the F-S or F-F supported strings higher than those of the S-S supported string.

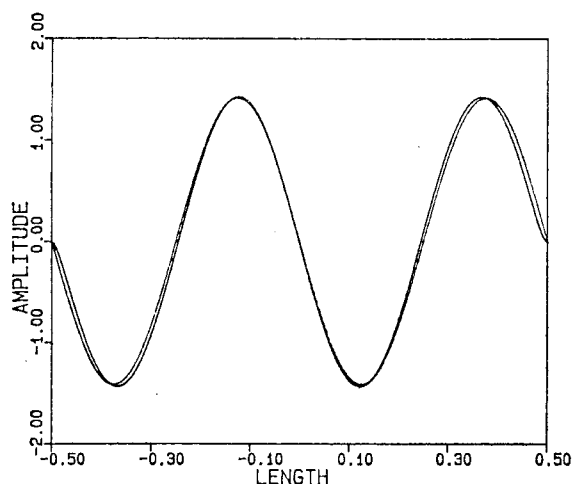


Fig. 1 Fourth mode shape of a stiff string with  $B=0.01$  and with the F-F support condition.

It seems that the lower modes are more affected by the reduction of the effective length, and consequently, the spacing between adjacent frequencies are smallest for the F-F support condition and largest for the S-S support condition.

### 3.2 Resonance Frequencies of a String with an Additional Mass

Once the resonance frequencies and modal shapes of a uniform stiff string are known, the resonance frequencies of a stiff string with an additional lumped mass can be calculated using Eq. (19). Instead of a representative value of  $B$ , practical values of a string used for middle  $C$  were chosen for the calculation. Those are shown in Table 4. The value  $B$  calculated from these parameters is  $0.3636 \times 10^{-3}$  which is fairly close to the value shown in reference.<sup>7)</sup> The resonance frequencies of a stiff string with the F-F support condition were calculated for two values of mass ratio,  $C=0.1$  and  $0.05$ . For  $C=0.1$ , the ratio of the distance of the mass from the end of the string to the total length of the string ( $\Delta l/l$ ) was chosen such that  $\Delta l/l=0.0, 0.01, 0.015$ , and  $0.02$ . If  $\Delta l/l=0$ , it is equivalent to the case with no additional mass. For  $C=0.05$ , two values of  $\Delta l/l$  were selected for calculation such that  $\Delta l/l=0.015$  and  $0.02$ . The results are shown in Fig. 2 and Fig. 3 for  $C=0.1$  and  $0.05$ , respectively. The frequency deviations are shown by the inharmonicity defined below:

$$\text{Inharmonicity} = 1,200 \log(f_n/nf_1) \log_2(\text{cents}) \quad (20)$$

Obviously, a purely harmonic series has zero

Table 4 Parameters of a string used for middle  $C$ .

Young's modulus	$E$	$19.5 \times 10^{11}$ dyn/cm <sup>2</sup>
Diameter	$D$	0.1042 cm
Linear density	$\mu$	0.0672 g/cm
Length	$l$	63.88 cm
Resonance frequency	$f_0$	261.6 Hz

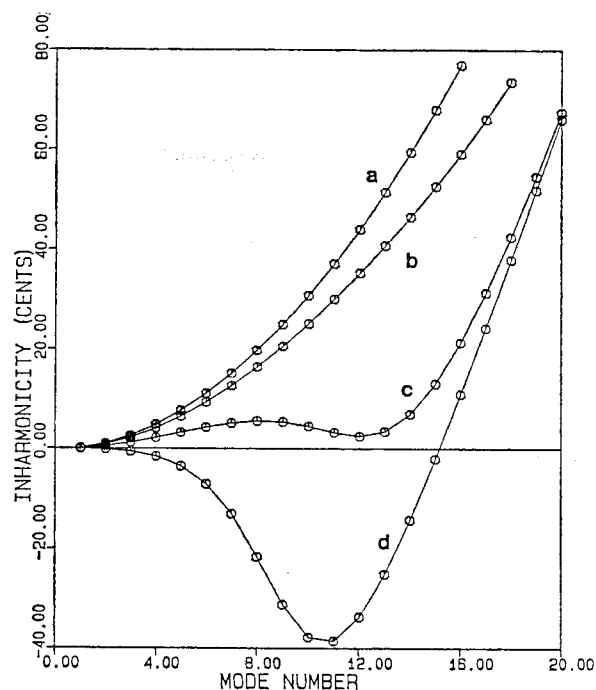


Fig. 2 Inharmonicities of string with an additional mass. The curves a, b, c and d are for  $C=0.01$  and  $\Delta l/l=0.0, 0.01, 0.015$  and  $0.02$ , respectively.

inharmonicity for any mode number  $n$ .

Curve (a) in Figs. 2 and 3 shows that the inharmonicity of a middle  $C$  string without an additional mass approaches approximately 120 cents (=semitone) for the 20th partial when the string is fixed-supported at both ends. For  $C=0.1$  and  $\Delta l/l=0.02$  (curve (d) in Fig. 2), the mass overcompensates the effect of the string stiffness and the inharmonicity becomes negative for mode numbers  $n=2 \sim 15$ . In case of  $C=0.1$  and  $\Delta l/l=0.15$ , the inharmonicity is very small for  $n \leq 15$ . Even for the higher mode numbers, the inharmonicity of this string is much smaller than that of a string without an additional mass. Curve (f) in Fig. 3 shows that even a mass having 5% of the string mass reduces the inharmonicity significantly when it is attached near

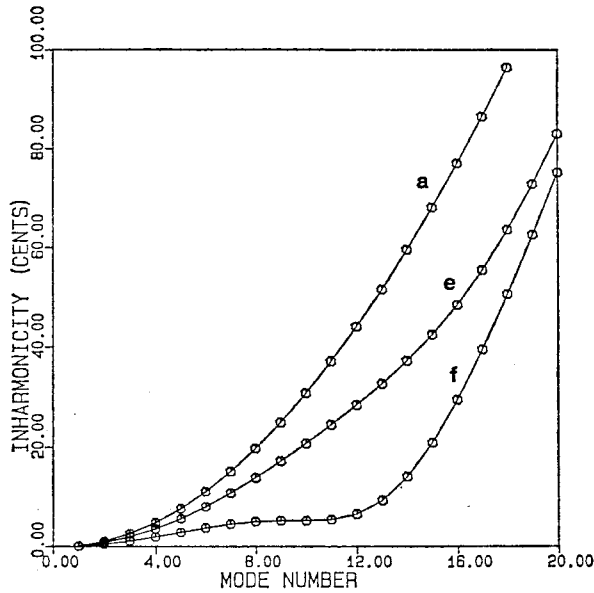


Fig. 3 Inharmonicities of a string with an additional mass. The curves a, e, and f are for  $C=0.05$  and  $\Delta l/l=0.0, 0.015$ , and  $0.02$ , respectively.

the end at a point 2% of the string total length. Figures 2 and 3 indicate that a proper combination of parameters around  $C=0.1 \sim 0.05$  and  $\Delta l/l=0.015 \sim 0.02$  may give a reasonable reduction of inharmonicity for the strings used around the key number 40 (middle C).

### 3.3 Comparison of Theoretical and Experimental Results

Figure 4 shows one end of an experimental setup used to measure the resonance frequencies of a string. The string is attached to a massive steel block through a holder and a force transducer. The holder gives the fixed condition to the string. The other end of the string is fixed by a holder alone. The transducer picks up the force caused by the vibration of the string. The length, the diameter, and the linear density of the string are given in Table 4. The tuning screw enables one to adjust the tension of the string as desired. The fundamental resonance frequency was adjusted as close as possible to 261.6 Hz (middle C). Then, the resonance frequencies up to the 20th mode were measured with and without the additional mass. The mass ratio  $C$  and the length ratio  $\Delta l/l$  were 0.09 and 0.016, respectively.

The results are shown in Figs. 5 and 6 for the strings without and with the additional mass, respectively. The lines with circles are the theoretical

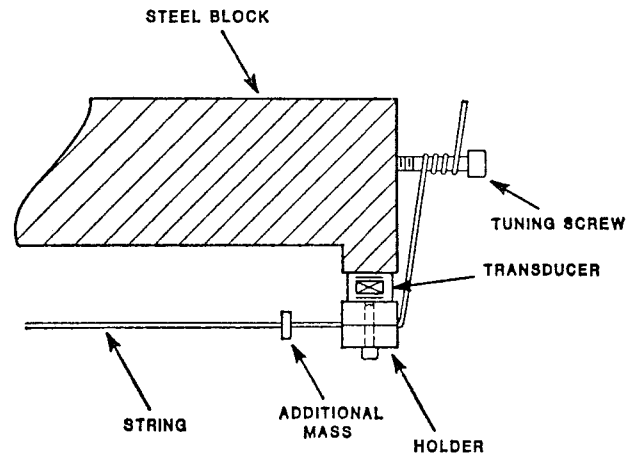


Fig. 4 Experimental setup for the measurement of resonance frequencies of a string with F-F support condition.

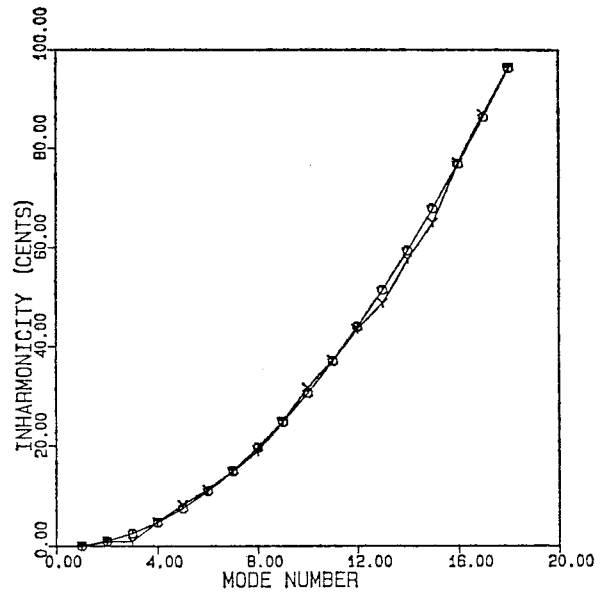


Fig. 5 Theoretical (circle) and experimental (Y) inharmonicities of a string with and additional mass ( $C=0.09$  and  $\Delta l/l=0.16$ ). The parameters of the string are given in Table 4.

results and the lines with Y are the experimental results. The theoretical and the experimental results of the string without the additional mass (Fig. 5) are in excellent agreement with each other. The results of the string with additional mass (Fig. 6) are also in good agreement. The experimental results show that the effect of the additional mass is slightly more than predicted. One possible reason for this is that the rotary inertia of the attached mass is neglected in the theory. It can be

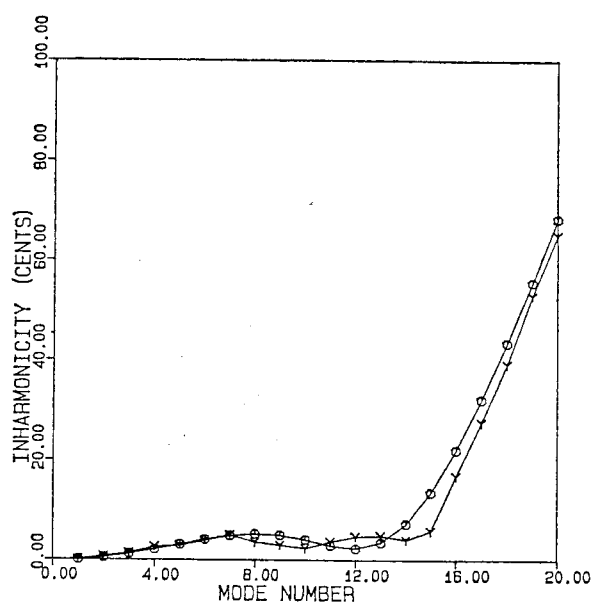


Fig. 6 Theoretical (circle) and experimental (Y) inharmonicities of a string with an additional mass ( $C=0.09$  and  $\Delta l/l=0.16$ ). The parameters of the string are given in Table 4.

concluded, however, that the theory can predict the experimental results very well.

#### 4. CONCLUSION

At first, the theories to calculate the resonance frequencies of a stiff string with and without an additional mass were presented. The resonance frequencies of a stiff string for three representative values of  $B$  ( $B=0.0001$ ,  $0.001$ , and  $0.01$ ) were calculated for three different support conditions (S-S, S-F, and F-F). The approximate formula to calculate the resonance frequencies of a stiff string given in Refs. 2) and 7) was found to be quite accurate for low to mid values of  $B$ . The formula, however, does not hold for larger values of  $B$  ( $\geq 0.01$ ). In this region, the inharmonicity is largest for the S-S support and smallest for the F-F support. For smaller  $B$ , the inharmonicities are the same for the three support conditions.

The effects of an additional mass were calculated for a string with parameters corresponding to those

of the key number 40 (middle C). An additional mass with a weight of 5~10% of the string weight, when attached at a place 2~1.5% of the total length of the string from one end, worked well to reduce the inharmonicity of the string. An experimental setup was made to measure the resonance frequencies of a real string with the F-F support condition. The agreement between the theoretical and experimental results was very good both for the strings with and without an additional mass.

As F. Miller, Jr.<sup>6)</sup> suggested, it seems very important to make a complete piano with much less inharmonicity from the lowest to the highest key. This will provide us with precious information about how the inharmonicity affects the timber of each tone, the octave stretching, and the quality of a piano as a whole.

#### ACKNOWLEDGMENTS

This study was conducted while the author was at CBS Technology Center in Stamford, CT, from 1981 to 1985. The author would like to acknowledge the helpful daily discussions he had with his colleagues, Gordon Ebit and William Strong. He further wants to express his gratitude to Junko Asaka at Ono Sokki Co., Ltd., Japan, for her typing and editorial work.

#### REFERENCES

- 1) Lord Rayleigh, *The Theory of Sound* (Macmillan and Co., Ltd., London, 1894).
- 2) P. M. Morse, *Vibration and Sound* (McGraw-Hill Book Co., Inc., New York, 1948).
- 3) R. S. Shankland and J. W. Coltman, "Departure of overtones of a vibrating wire from a true harmonic series," *J. Acoust. Soc. Am.* **10**, 161-166 (1939).
- 4) O. H. Schuck and R. W. Young, "Observations on the vibration of piano strings," *J. Acoust. Soc. Am.* **15**, 1-11 (1943).
- 5) R. W. Young, "Inharmonicity of plain wire piano strings," *J. Acoust. Soc. Am.* **24**, 267-273 (1952).
- 6) F. Miller, Jr., "A proposed loading of piano strings for improved tones," *J. Acoust. Soc. Am.* **21**, 318-322 (1949).
- 7) H. Fletcher, "Normal vibration frequencies of a stiff piano string," *J. Acoust. Soc. Am.* **36**, 203-209 (1964).