

INHARMONICITY OF A STEINWAY
MODEL L GRAND PIANO
AND ITS EFFECT ON TUNING

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I. INTRODUCTION

The effect of the inharmonicity of piano strings on the tuning of the octave (F3-F4) was discussed in the Interim Report, "REVIEW OF THE TUNING PROCEDURE TAKING INTO ACCOUNT THE INHARMONICITY OF A PIANO STRING". In that report, the inharmonicity indexes (B) of the strings were assumed to be equal to 0.0004 throughout the range F3-F4. This was done to roughly approximate the inharmonicity characteristic in the F3-F4 octave of an upright piano which was used by Fletcher in his study (JASA, 36, 203-214, 1964). It was shown that the theoretically calculated beat rates can never be achieved when applied to a piano with the above mentioned inharmonicity characteristic. The inharmonicity characteristics of Steinway pianos have only been reported by Young (JASA, 24, 267-273, 1952). He also published a paper discussing the effect of inharmonicity on tuning (JASA, 5, 1-11, 1943).

For this reason, the inharmonicity characteristic of the Model "L" piano at CTC was measured, and its effect on tuning was discussed. This report will describe these results and also discuss a new method to calculate beat rates of a smoothly stretched octave in a specific piano with a known inharmonicity characteristic.

II. INHARMONICITY MEASUREMENT

A theoretical (and simplified) formula to calculate the resonance frequencies of the partials of a stiff string is given by

$$f_n = F \cdot n(1 + 1/2Bn^2), \quad n = 1, 2 \dots \quad (1)$$

and

$$B = (\pi^2 ESR^2 / Tl^2) \quad (2)$$

where, F is a constant, f_n is the resonance frequency of the n th mode, and B is the inharmonicity index determined by the Young's modulus E , cross sectional area S , radius of gyration R ($R = d/4$ with string diameter d), tension T , and length l . To determine the inharmonicity of a string, at least two frequencies for different n must be measured since there are two unknowns in Eq. (1) (F and B). If f_m and f_n are the measured resonance frequencies of the m th and n th modes, respectively, it follows from Eq. (1) that

$$F^2 = \{[(m/n)f_n]^2 - [(n/m)f_m]^2\}, \quad (3)$$

and

$$B = [(f_n/f_m)^2(m/n)^2 - 1] / [n^2 - (f_n/f_m)^2(m/n)^2m^2] \quad (4)$$

To measure the resonance frequencies of each note, an accelerometer is attached to the bridge as closely as possible to the strings of the note. The Structural Dynamics Analyzer HP5423A starts to analyze the signal from the accelerometer with some time delay after the key is struck. The time delay was set approximately 0.5 sec or larger for low-frequency notes, and 0.2 sec for high-frequency notes. Using this method, the resonance frequencies of the string should be accurately measured even though the magnitudes of the partials are affected by the vibration characteristics of the soundboard.

In general, however, measuring the resonance frequencies with good precision is not an easy task. The possible reasons are:

- (1) The durations (in other words, decay rates) of the partials are different for different mode numbers.
- (2) The frequency of each partial seems to vary slightly as a function of the time from the initial attack.
- (3) Some partials show a blunt peak due either to the limited analysis capability of the analyzer or due to the overlap of the string and the soundboard resonances.
- (4) Extraneous resonances exist, probably, due to the resonances of the adjacent strings even though they are damped by a felt strip and foot-pedal dampers. Another possible cause of the extraneous resonances is the longitudinal vibration of the string. (Baldwin Piano Company has a patent of a wound string which is supposedly free from this problem, Synchro Tone Strings, U.S. Pat. No. 3, 523, 480.)
- (5) The resonance frequencies of partials may not obey Eq. (1) since the boundary condition at the bridge is not ideal.

A careful measurement, with attention to the above mentioned problems, is necessary to obtain a fairly consistent value of B for each note. For low- to mid-frequency notes, it is possible to measure f_n for n up to 15 or 20. For high-frequency notes, this is limited to $n \leq 10$ because of low signal-to-noise ratios. When the frequencies of n partials are measured, there are $n(n-1)/2$ possible pairs. Each pair gives the values of F and B using Eqs. (2) and (3). The averaged F and B can be used as an estimate of F and B . If there

are any measured partial frequencies which do not seem reliable, these frequencies should be excluded from the averaging. Since the lower order partial frequencies deviate very slightly from the harmonic relationship, a same frequency-measurement error causes a larger error in the estimation of B when the modes of lower orders are used especially for small B. If this is the case, it is recommended not to use several lowest modes for the estimation of B.

Another convenient formula derived from Eq. (1) is given by

$$f_n = nf_1[1 + 1/2 B(n^2-1)] \quad (5)$$

This formula describes the relationship between the fundamental frequency f_1 and the higher partial frequencies f_n .

III. INHARMONICITY OF A MODEL L GRAND PIANO

The measured inharmonicity index B of the Model L Grand Piano at CTC is shown by circles in Fig. 1. The keyboard is used as the horizontal coordinate showing the notes of the inharmonicity measurement. Figure 1 also shows data of the upright piano in the Fletcher's paper. The transition from wound strings to solid strings occurs between key number 26(A#3) and 27(B3) for the Model L Grand Piano. The same thing seems to happen around key number 31(D#3) for the upright piano. Below key number 10, the grand piano has lower B than the upright except for a couple of the lowest keys. Between key numbers 10 and 30, both the grand and the upright pianos have approximately the same inharmonicity. The upright piano has a big jump when the strings change from a wound to a solid type. On the other hand, the grand piano gradually increases the inharmonicity around the transition from wound to solid strings. Around key number 42, both pianos have approximately the same

inharmonicities. It is surprising that the grand piano has a larger inharmonicity than the upright piano above key number 42.

The inharmonicity of note C4 is approximately equal to the value calculated from Eq. (2). Especially for the solid strings, Eq. (2) seems valid and can be used to estimate inharmonicities instead of actually measuring the partial frequencies and calculating B using Eqs. (3) and (4). The reason for the larger inharmonicities of the mid to treble notes of the grand piano may be that it uses a shorter or a thicker string than the upright piano for the same note.

The inharmonicity characteristic in the F3-F4 octave is shown in detail in Fig. 2. In this range, the inharmonicity characteristic seems to be well approximated by a straight line (shown by the dotted line in Fig. 2).

If the integer numbers from 1 to 13 are assigned for each note from F3 to F4, the inharmonicity index B(N) is given by

$$B(N) = B_1 \cdot 10^{\frac{(N-1)}{12} \log(B_{13}/B_1)} \quad (6)$$

where B_1 and B_{13} are the assumed inharmonicity indexes of F3 and F4, respectively. The values $B_1=0.18 \times 10^{-3}$ and $B_{13}=0.6 \times 10^{-3}$ correspond to a straight line in Fig. 2. The approximated inharmonicity characteristic of Eq. (5) will be used to discuss the effect of inharmonicity on tuning in the next section.

IV. EFFECT OF INHARMONICITY ON TUNING

The difficulty with aural tuning is that a tuner must tune one note to another already tuned note using a beat rate assigned to this specific tuning interval. If the beat rate were always zero, the tuning might be easier since he would not have to count the beat rate. For aural tuning, however, this is only the case when octave intervals are tuned. (Even in this case, a perfect zero-beat tuning is impossible because of the inharmonicity.) Another difficulty with aural tuning is that there is more than one beat when two notes of a tuning interval are played at the same time. Electronic tuning, on the other hand, is free from both of these problems. Thus, if the piano tone does not have the inharmonicity property, the electronic tuning may work perfectly well. The problem with the electronic tuning is probably that the octave stretching is necessary to properly tune an actual piano. Present-day electronic tuning devices are not designed to completely accommodate the stretched octave, even though they have an adjustment knob to slightly sharpen or flatten the pre-set frequencies. Here, a method of determining the frequencies of notes within the F3-F4 octave will be described which are gradually shifted to match the stretched F3-F4 octave. The beat rates calculated from these frequencies can be used for the aural tuning as well as the electronic tuning.

First of all, the note C4 is tuned to a tuning fork (or an electronic tuning device) with a standard frequency of 523.252Hz. Because of the inharmonicity of C4, the fundamental resonance frequency of this note is slightly lower than $523.252/2 = 261.626\text{Hz}$. In the present case, this is given as 261.485Hz (assuming that the inharmonicity index of C4, B8, is equal to 0.00036 from Fig. 2 or Eq. (6)) which is 0.93 cents lower than 523.252Hz.

The next step is to determine the note frequencies in the range F3 to F4 with the reference frequency C4 unchanged. The frequency ratios of F4/F3 and F5/F4 are given by (using Eq. 5))

$$X_1 = 2(1+1.5B_1)$$

and $X_2 = 2(1+1.5B_{13})$

respectively. If B_{13} is larger than B_1 (mostly this is true), the F4-F5 interval is stretched more than the F3-F4 interval. It seems appropriate to increase the frequency ratio of adjacent notes gradually to match the stretched intervals of F3-F4 and F4-F5. The condition that the frequency ratio of adjacent notes increases by a constant from F3 to F5 is enough to determine the frequencies from F3 to F4. The frequencies of note F#4 and higher need not be determined at this stage, but this condition may ensure a smooth transition of the frequency ratio of adjacent notes from E4-F4 to F4-F#4. Figure 3 shows the frequency ratios of adjacent notes that satisfy the above condition. The initial frequency ratio of F#3/F3 is b . The next frequency ratio (G3/F#3) increases by a factor of a . Thus, it is equal to ab . The following frequency ratio is a^2b , and so on. From these frequency ratios of adjacent notes, the frequency ratios of F4/F3 and F5/F4 are given by $a^{66}b^{12}$ and $a^{210}b^{12}$, respectively. Therefore, we have the relationship:

$$a^{66}b^{12} = 2(1 + 1.5B_1) \tag{7}$$

and $a^{210}b^{12} = 2(1 + 1.5B_{13}) \tag{8}$

Dividing Eq. (8) by Eq. (7), we have

$$a^{144} = 1 + 1.5(B_{13} - B_1)$$

where $B_{13}, B_1 \ll 1$ is assumed. Since $(B_{13} - B_1)$ is very small, a is approximately given by

$$a = 1 + 0.010417(B_{13} - B_1) \quad (9)$$

Substituting Eq. (9) into Eq. (7) we have

$$b^{12} = 2(1 - 0.6875 B_{13} + 2.1875 B_1)$$

Again, since B_{13} and B_1 are very small, b is approximately given by

$$b = 2^{1/2}(1 - 0.057292 B_{13} + 0.182292 B_1) \quad (10)$$

Using these two constants and starting from the reference frequency of C4, all other frequencies from F3 to F4 are determined. For example, the frequency of B4 is given by $f_c/(a^6b)$, where f_c is the reference frequency of C4. The frequency of A#4 is given by $[f_c/(a^6b)]/(a^5b)$, and so on. The frequency of C#4 is given by $f_c \cdot a^7b$.

Once the frequencies of the 13 notes from F3 to F4 are determined, the beat rates which can be used for the aural tuning are obtained. The result is shown in Table I for the case when the inharmonicity characteristic is given by the dotted line of Fig. 2. For example, the beat rate between F3 and C4 is 0.64/sec. The negative sign indicates that the high note (C4 in this case) has a partial frequency that is lower than the corresponding partial frequency of the low note (F3 in this case). In Table II, the beat rates of the ideally harmonic case are shown for comparison. The beat rates of the 5th intervals (F3-C4, F#3-C#4, ...) of the present case is slightly higher than the ideal case. The beat rates of 4th intervals (F3-A#4, F#3-B3, ...) in the present case is slightly lower than the ideal case. The beat rates of 3rd and 6th intervals in the present case increase smoothly within like intervals. This property is almost always used as a check for proper tuning. Generally speaking, it seems there is not much difference between Table I and Table II.

If the beat rate of Table II is used to tune a piano with the inharmonicity characteristic of the present case, however, an unevenness in the beat rates results. Table III shows the result. The double boxed numbers are the beat rates used for tuning. The 4th intervals F#3-B and C4-F4 that are not used for tuning have much smaller beat rates than the other 4th intervals. The 3rd and 6th intervals have uneven changes of beat rates within the like intervals. This is the problem with the theoretical beat rates, calculated from the ideally harmonic case, when they are used to tune a real piano. The numbers shown by the right-side columns of Tables I and III are the frequency shifts (in cents) of notes from the ideal case. The column of Table I shows that the interval of the adjacent notes increases smoothly from low notes to high notes as originally intended. On the other hand, the frequency shifts of Table III are rather random indicating an improper tuning. The frequency shifts in cents shown in Table I can be used in the electronic tuning if the device is able to shift the frequencies in 1/10 cent increments.

When a constant inharmonicity of 0.0004 is assumed throughout the F3-F4 octave (this was the case discussed in the last report), the beat rates calculated from the suggested procedure are significantly different than those of the ideal case. Table IV shows the result. In this case, the 4th intervals are almost beatless. The 5th intervals have higher beat rates than the ideal case. As Eq. (9) shows, the constant a is equal to unity in this case since $B_1 = B_{13}$ and, therefore, the frequency ratios of adjacent notes are all equal. However, this ratio ($=b$) is slightly larger than $2^{1/2} = 1.059463$. If the theoretical beat rates of Table II are applied in this case, the tuner may encounter more difficulty in tuning than in the previous case. The result is shown in Table V. The unevenness of beat rates is significantly large and the frequency shifts of notes from the ideal case is random and large.

As the above discussion shows, the suggested method to determine the frequencies of notes from F3 to F4 and, consequently, to determine the modified beat rates which can be used to tune pianos with stretched octaves seems to be appropriate. Obviously, the modified beat rates become more effective when tuning a piano with larger inharmonicity characteristic.

V. CONCLUSION

The inharmonicity characteristic of a Steinway Model L grand piano was measured. It was surprising that the 6-foot grand piano has larger inharmonicities in the treble region than the upright piano which was used in the Fletcher's study. The reason for this is probably due to the shorter or thicker string of the Model L grand piano.

A new method was presented to determine the frequencies of notes within the F3-F4 octave when the inharmonicity characteristic in the same region is known. This method takes care of the stretched octave F3-F4 by increasing the frequency ratio of two adjacent notes from $2^{1/2}$. Furthermore, this method makes it possible to gradually increase the frequency ratio of adjacent notes for a smooth transition of frequency ratios around F4. This smooth transition seems important since the F4-F5 interval is mostly wider than the F3-F4 interval. The modified beat rates obtained by this method seems to comply with the commonly used tests in the conventional tuning technique. The validity of applying the modified beat rates to the tuning of a real piano relies on whether each partial frequency of note obeys Eq. (1) or not. On an average basis, however, the modified beat rates may make the tuning easier since it has already taken into account the effect of the inharmonicity.

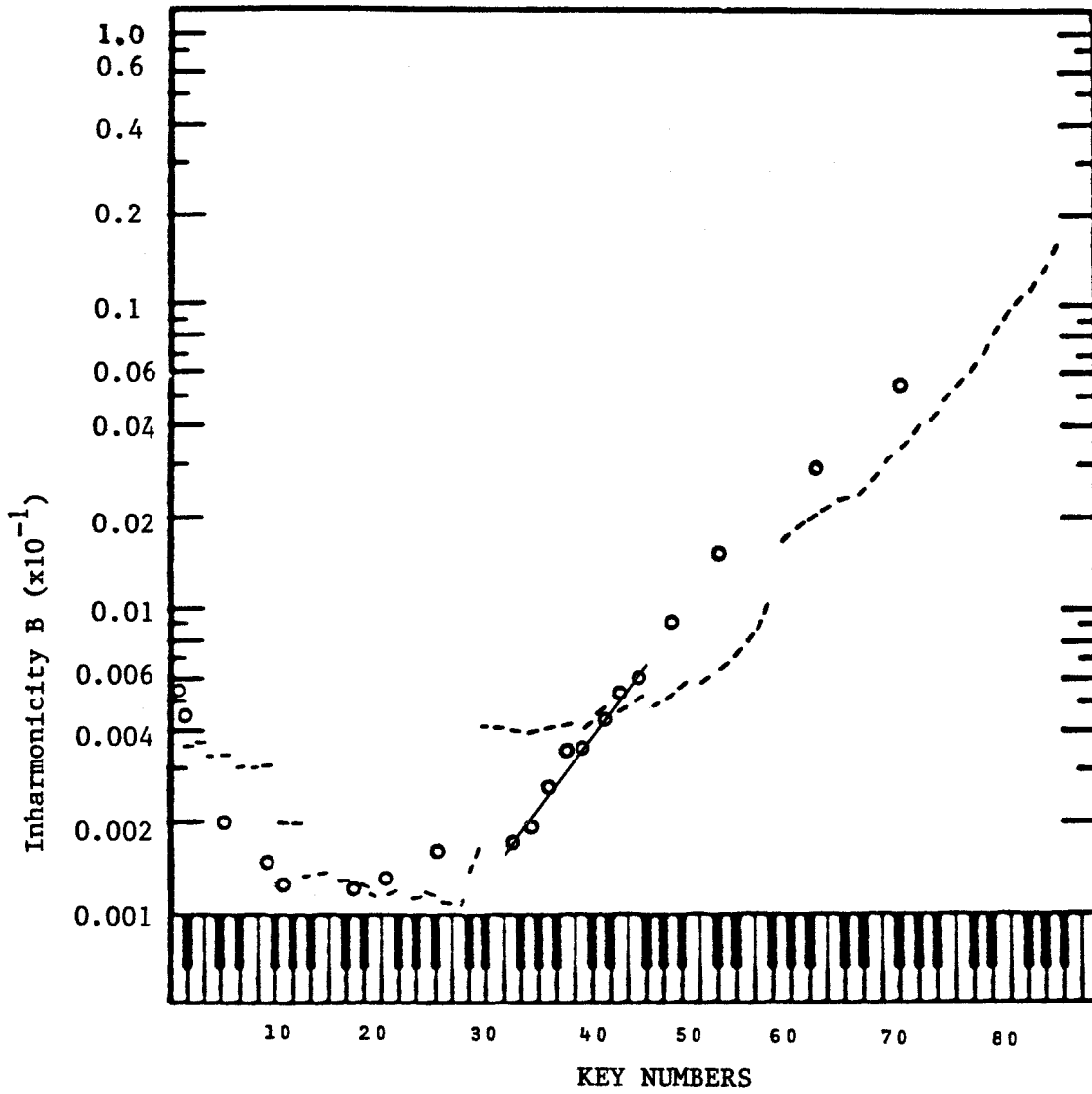
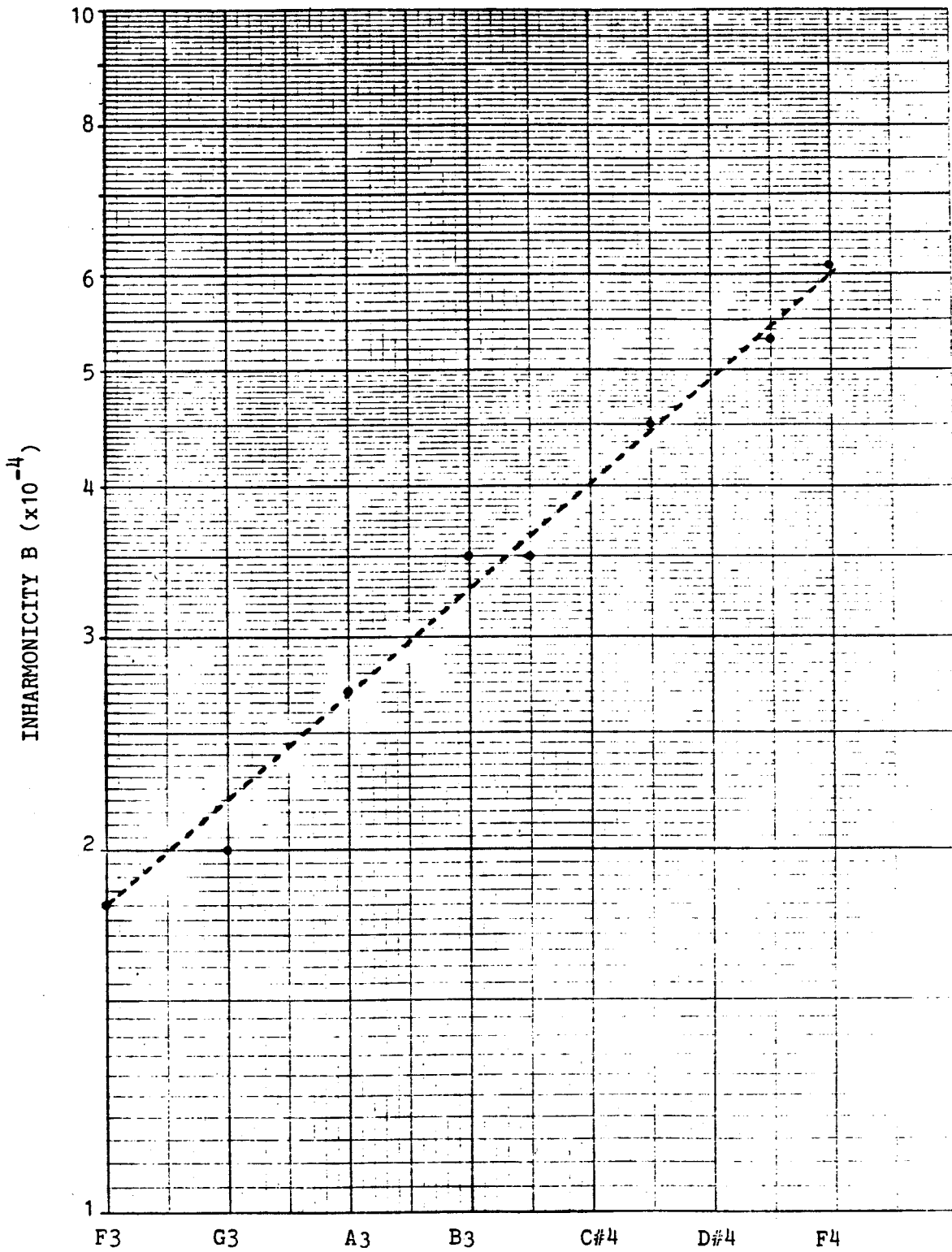


Fig. 1 Inharmonicities of a Steinway Model L Grand Piano (circles) and an upright piano (dotted lines, after Fletcher)



NOTE

FIG. 2 Measured (.) and approximated (dotted line) inharmonicities of a Steinway Model L grand piano in the temperament octave F3-F4.

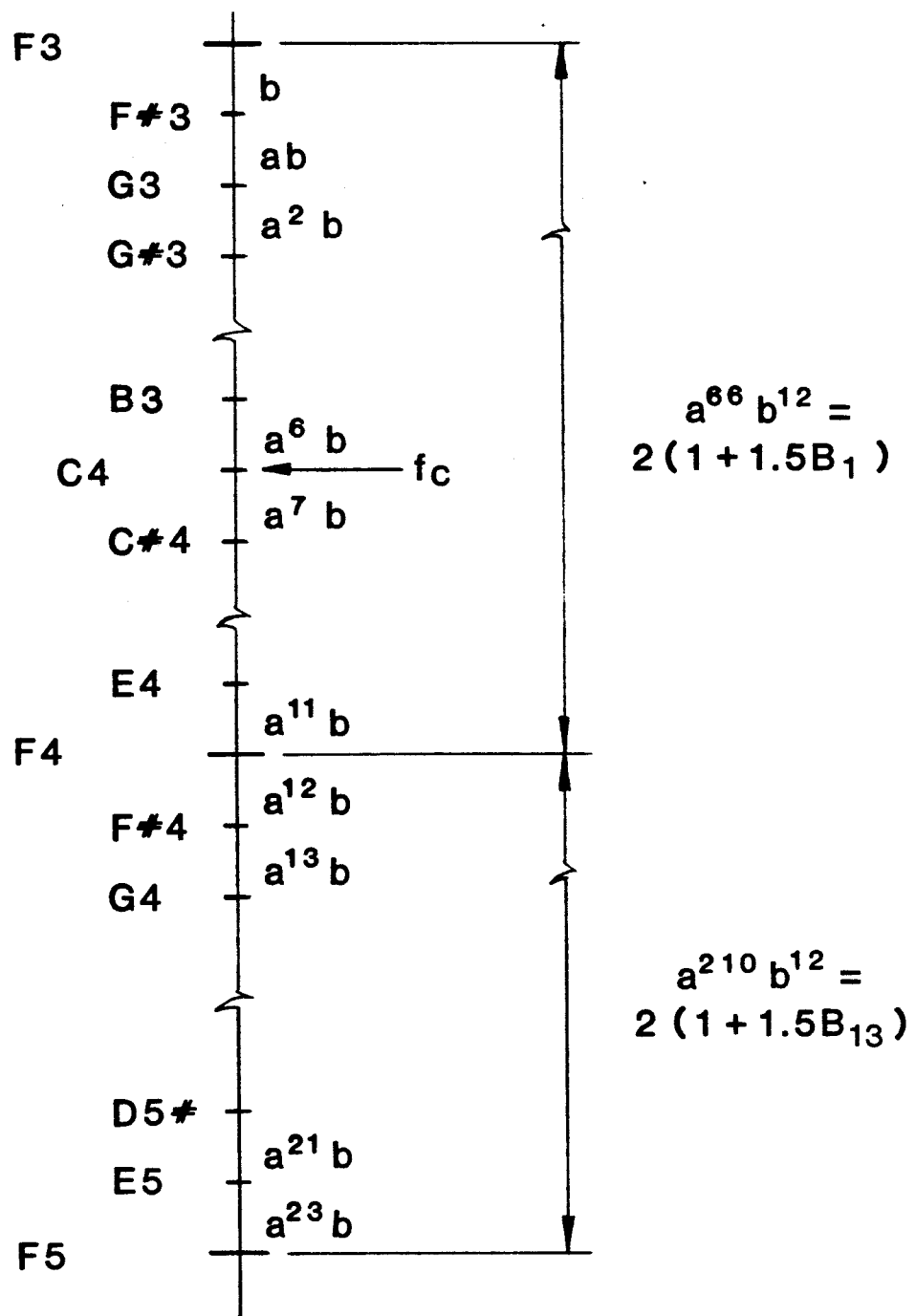


Fig. 3 Scale to show the frequency ratios of adjacent notes in the interval F3-F5.

CENTS DEV.

	F	G	A	A#/B ^b	B	C	C#/D ^b	D	D#/E ^b	E	F
F#G ^b	•	•	•	•	•	•	•	•	•	•	•
F	•	•	•	•	•	•	•	•	•	•	•
	-8.39	-10.91	-10.90	-13.43	-12.30	-13.09	-13.91	-15.95	-19.22	-19.23	-19.23
	6.97	6.86	6.84	6.84	7.66	7.66	8.54	8.57	8.57	8.43	8.43
	0.79	0.27	0.89	0.66	0.94	0.94	1.05	1.12	1.12	0.228	0.228
	-0.59	-0.27	0.89	0.66	0.94	0.94	1.05	1.12	1.12	0.228	0.228
	7.96	7.91	7.96	7.96	7.96	7.96	7.96	7.96	7.96	7.96	7.96
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	-0.78	-1.27	-0.23	-1.13	-0.20	-1.23	-3.08	-1.80	-2.42	-1.18	-1.84
	-0.62	-1.25	-0.62	-1.25	-0.62	-1.25	-0.62	-1.25	-0.62	-1.25	-0.62

Table III. Beat rates of intervals and cents deviation of each note (F3-F4) for B₁₂=0.00018, B₈=0.00036, and B₁₃=0.0006 when the tuning steps C4-F3(-F4)-A#4-D#4-G#4-C#4-F#3 and C4-G3-D4-A4-E4-B4 are taken using the beat rates of ideally harmonic case (double-boxed numbers).

CENTS DEV.

	F	G	A	B	C	D	E	F
F#G#
F	-9.88	-15.73	-11.09	-17.66	-12.43	-4.95	-23.61	-19.76
G
A#B#
A	4.68	4.98	5.25	5.59	11.17	6.26	7.02	-1.66
B
C#D#
C	-0.59	-0.62	-0.89	-0.94	1.00	1.05	1.12	-2.04
D
D#E#
E
F#G#
F	0	0	0	0	0	0	0	0

Table V. Beat rates of intervals and cents deviation of each note (F3-F4) for $B_1 = B_8 = B_{13} = 0.0004$ when the tuning steps C4-F3 (-F4)-A#4-D#4-G#4-C#4-F#3 and C4-G3-D4-A4-E4-B4 are taken using the beat rates of ideally harmonic case (double-boxed numbers).