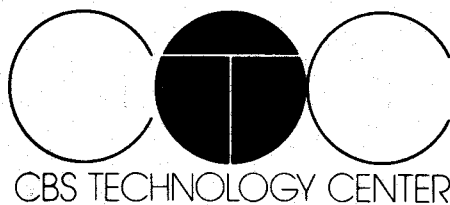


CONTROL OF THE INHARMONICITY OF
A STIFF STRING
BY THE ADDITION OF A LUMPED MASS

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LIST OF SYMBOLS

A_n	coefficient of mode function $Y_n(x)$
B	parameter defining the inharmonicity of a string (Eq. (7))
C	mass ratio $M_a / \ell\mu$
D	diameter of a string ($= 2R$)
E	Young's modulus of a string
$K_1 - K_4$	coefficients defined by Eq. (9)
L	operator defined by Eq. (3)
M_a	additional lumped mass
R	radius of a string
S	area of a cross-section of a string ($= \pi R^2$)
T	tension of a string
$Y(x)$	displacement function of a string
$Y_n(x)$	n th mode shape of a string
f_n	n th resonance frequency of a string
f_0	fundamental resonance frequency of an ideal string
j	unit of imaginary number $\sqrt{-1}$
k, k_1, k_2	wave numbers
ℓ	length of a string
$\Delta\ell$	distance of the position of the lumped mass from the near end
m, n	mode numbers
t	independent time parameter
x	coordinate in the length direction of a string
x_0	position of the additional mass

y	time dependent displacement function
$\delta(x)$	delta function
μ	linear density
π	circular constant
ω	angular frequency
ω_n	nth mode angular frequency

ABSTRACT

A method for controlling the inharmonicity of a stiff string by adding a lumped mass (or masses) was originally suggested by F. Miller, Jr. 38 years ago.⁶ In the present paper, the same idea is reviewed in a more precise manner. The effect of the stiffness of the string on its inharmonicity is discussed for different support conditions (the simple-simple, simple-fixed, and fixed-fixed conditions). The simple-simple support condition gives the largest inharmonicity (deviation from a true harmonic series) among them even though the differences are small for most of the cases. The theoretical and experimental results of the resonance frequencies of a string with the fixed-fixed condition show very good agreement with each other. For a middle C string, a mass of 0.2 - 0.4 gr when attached 1.3 - 0.9 cm from one end of the string reduces the inharmonicity significantly.

0 INTRODUCTION

It is well known that the frequencies of the partials of a piano tone are not integer multiples of the fundamental resonance frequency. Both the theoretical and experimental aspects of the vibration of a uniform stiff string have been investigated.¹⁻⁷ Lord Rayleigh¹ and A.P.M. Morse² contributed to the theoretical aspect of the vibration of a stiff string. R.S. Shankland and J.W. Coltman measured the resonance frequencies of a stiff string and compared them with the theoretical results obtained by Lord Rayleigh. O. H. Schuck and R. W. Young⁴ measured the inharmonicity (deviation from a true harmonic) of piano tones for different key numbers and different types of pianos, and showed that a large (grand) piano has less inharmonicity than a smaller piano. They also showed that the inharmonicity causes deviations in tuning (octave stretching). R.W. Young⁵ furthermore measured six pianos of various styles and makes, and raised a hypothesis that those pianos are "best" in which the inharmonicity is least and in which it changes smoothly. F. Miller, Jr.⁶ discussed the possibility of controlling the inharmonicity by adding a lumped mass near the end of a string. H. Fletcher⁷ covered both the theoretical and experimental aspects of this problem.

The idea suggested by F. Miller, Jr.⁶ seems very attractive since it makes it possible to control the inharmonicity of a stiff string by adding a small mass near one end of the string. The addition of the lumped mass lowers the resonance frequency of each partial. For the lower order partials, the effect of the mass is small since the mass is close to the nodes of the normal vibration of the string. As the order of the partial increases, the position of the anti-node approaches the position of the mass and, therefore, the mass works more effectively to lower the resonance frequency.

The present paper describes at first the theories used to calculate the resonance (eigen) frequencies and mode shapes of a uniform stiff string with and without the additional mass. Then, the resonance frequencies of a stiff string with different degrees of inharmonicity for the simple-simple, simple-fixed, and fixed-fixed conditions are numerically calculated without approximation. Finally, the theoretical and experimental results of the resonance frequencies of a stiff string with an additional mass of different sizes and at different positions on the string are compared using an experimental setup which gives an ideal fixed-fixed support condition.

L MATHEMATICAL DISCUSSION

1.1 Resonance Frequencies and Mode Shapes

The basic equation of motion of a homogeneous stiff string is⁷

$$ESR^2 \frac{\partial^4 y(x,t)}{\partial x^4} + \mu \frac{\partial^2 y(x,t)}{\partial t^2} - T \frac{\partial^2 y(x,t)}{\partial x^2} = 0. \quad (1)$$

For a steady state motion, the time dependent term $e^{j\omega t}$ can be avoided using $y=Y(x) e^{j\omega t}$, such as

$$ESR^2 \frac{d^4 Y(x)}{dx^4} - T \frac{d^2 Y(x)}{dx^2} - \mu \omega^2 Y(x) = 0. \quad (2)$$

Define the operator L such that

$$L = \left(ESR^2 \frac{d^4}{dx^4} - T \frac{d^2}{dx^2} \right), \quad (3)$$

then

$$L \{ Y(x) \} = \mu \omega^2 Y(x). \quad (4)$$

Assuming a solution of the form $e^{2\pi kx}$ and substituting this into Eq. (2), we have

$$k^4 - (1/4Bl^2)k^2 - (f^2/16Bl^4f_0^2) = 0. \quad (5)$$

Solving Eq. (5), we have four solutions:

$$k = \pm k_1$$

$$\text{and } k = \pm jk_2,$$

where

$$k_1 = \left[\frac{1}{8Bl^2} \left\{ \left(1 + \frac{4Bf^2}{f_0^2} \right)^{\frac{1}{2}} + 1 \right\} \right]^{\frac{1}{2}} \quad (6a)$$

and

$$k_2 = \left[\frac{1}{8Bl^2} \left\{ \left(1 + \frac{4Bf^2}{f_0^2} \right)^{\frac{1}{2}} - 1 \right\} \right]^{\frac{1}{2}}. \quad (6b)$$

The notations

$$B = (\pi^2 ESR^2 / Tl) \quad (7)$$

and

$$t_0 = (\tau/\mu)^{\frac{1}{2}}/2l \quad (8)$$

were also used.

The general solution of Eq. (2) is then given by

$$Y(x) = K_1 \cosh 2\pi k_1 x + K_2 \sinh 2\pi k_2 x \\ + K_3 \cos 2\pi k_2 x + K_4 \sin 2\pi k_2 x. \quad (9)$$

The boundary conditions are expressed as follows.

For the fixed boundary condition,

$$\left. \begin{array}{l} Y(x) = 0 \\ \frac{dY(x)}{dx} = 0 \end{array} \right\} \text{at } x = -\frac{l}{2} \text{ or } \frac{l}{2}. \quad (10)$$

For the simply supported boundary condition,

$$\left. \begin{aligned} Y(x) &= 0 \\ \frac{dY(x)}{dx} &= 0 \end{aligned} \right\} \text{ at } x = -\frac{l}{2} \text{ or } \frac{l}{2}. \quad (11)$$

The four equations given by Eq. (10) and, or, Eq. (11) determines the resonance frequencies f_n and the corresponding mode shapes $Y_n(x)$. These results will be used to obtain the resonance frequencies and mode shapes of a stiff string with an additional lumped mass.

1.2 Eigenfrequencies and Eigenfunctions of a String with an Additional Lumped Mass

Assuming that the resonance frequencies and mode shapes of a uniform string are known, the deflection of the string with an additional lumped mass is expressed by

$$Y(x) = \sum_{n=1}^{\infty} A_n Y_n(x), \quad (12)$$

where

$$L \{Y_m(x)\} = \mu \omega_m^2 Y_m(x) \quad (13)$$

and

$$\int_{-l/2}^{l/2} Y_m(x) Y_n(x) dx = l \delta_{mn}. \quad (14)$$

Eq. (14) is obtained using the orthogonal property of Eq. (12) and letting the norm of each function equal to l .

The equation of motion with the additional mass is

$$L \{Y(x)\} - \mu \omega^2 Y(x) = -j\omega M_a \delta(x-x_0) \cdot j\omega Y(x). \quad (15)$$

Substituting Eq. (12) into Eq. (15), we have

$$L \left\{ \sum_{n=1}^{\infty} A_n Y_n(x) \right\} - \mu \omega^2 \sum_{n=1}^{\infty} A_n Y_n(x) = M_a \omega^2 \delta(x-x_0) \sum_{n=1}^{\infty} A_n Y_n(x). \quad (16)$$

Multiplying by $Y_m(x)$ and integrating over $[-l/2, l/2]$,

$$\begin{aligned} L \left\{ \sum_{n=1}^{\infty} A_n \int_{-l/2}^{l/2} Y_n(x) Y_m(x) dx \right\} - \mu \omega^2 \sum_{n=1}^{\infty} A_n \int_{-l/2}^{l/2} Y_n(x) Y_m(x) dx \\ = M_a \omega^2 \sum_{n=1}^{\infty} A_n \int_{-l/2}^{l/2} Y_n(x) Y_m(x) \delta(x-x_0) dx. \end{aligned} \quad (17)$$

Using Eqs. (13) and (14),

$$l\mu(\omega_m^2 - \omega^2) A_m = M_a \omega^2 \sum_{n=1}^{\infty} A_n Y_n(x_0) Y_m(x_0). \quad (18)$$

Therefore,

$$\omega^2 \left\{ A_m + \left(\frac{M_a}{l\mu} \right) \sum_{n=1}^{\infty} A_n Y_n(x_0) Y_m(x_0) \right\} = \omega_m^2 A_m, \quad m = 1, 2, \dots. \quad (19a)$$

or in a matrix form:

$$\begin{bmatrix} \omega^2 \{1 + C Y_1^2(x_0)\} - \omega_1^2 & \omega^2 C Y_1(x_0) Y_2(x_0) & \dots & \dots \\ \omega^2 C Y_2(x_0) Y_1(x_0) & \omega^2 \{1 + C Y_2^2(x_0)\} - \omega_2^2 & \dots & \dots \\ \vdots & \vdots & \dots & \dots \\ \vdots & \vdots & \dots & \dots \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \end{Bmatrix}, \quad (19b)$$

where $C = M_a / \ell \mu$.

Eq. (19) indicates that the resonance frequencies and mode shapes with the additional mass are determined by the resonance frequencies and mode shapes of the homogeneous stiff string, the ratio of the additional mass to the string mass and the position of the additional mass on the string.

II. RESULTS AND DISCUSSION

2.1 Resonance Frequencies and Modal Shapes of a Homogenous String

Using Eq. (9) and the boundary conditions expressed by Eqs. (10) and/or (11), the resonance frequencies of a stiff string normalized to the fundamental resonance frequency of an ideal string (Eq. (8)) were calculated for three representative values of B . The results are shown in Table I - Table III. For convenience, the fixed and simply supported boundary conditions are expressed simply by "F" and "S", respectively. It is reasonable that the F-F support gives the highest resonance frequencies among three different kinds of support. For $B=0.001$ (which corresponds approximately to key number 60 of an upright piano, according to the reference [7]), the deviation of f_{20} from $20xf_1$ is approximately 290 cents (18.25%) for any type of support condition.

By looking at the tables carefully, the following can be concluded:

1. The resonance frequency of a stiff string for the S-F support is approximately the geometrical mean of the frequencies for the S-S and S-F supports.
2. Up to about $B=0.001$, the ratios between adjacent frequencies (frequency spacings) are the same for the three different kinds of supports.

3. For B larger than 0.001, the frequency spacing is widest for the S-S support and narrowest for the F-F support even though the differences are very small.

What properties (2) and (3) indicate is very interesting. Contrary to intuition, the S-S support condition gives the same or a larger inharmonicity than the other two support conditions do. The observation of the modal shapes may give a hint to this reason.

The modal shapes of a stiff string with the S-S support is exactly the same as the modal shapes of an ideal string⁷. The fixed boundary condition at one end or at both ends causes the modal shapes to deviate from those of the ideal string. As an example, the 4th mode of a stiff string with the F-F support and $B=0.001$ is shown by the curve with smaller slopes at both ends in Fig. 1. The other curve in the figure is the modal shape of the ideal string. It can be seen that the fixed boundary condition reduces the effective length of the string. This makes the resonance frequencies of the F-S or F-F supported strings higher than those of the S-S supported string. It seems that the lower modes are more affected by the reduction of the effective length, and consequently, the spacing between adjacent frequencies are smallest for the F-F support condition and largest for the S-S support condition.

2.2 Resonance Frequencies of a String with an Additional Mass

Once the resonance frequencies and modal shapes of a uniform stiff string are known, the resonance frequencies of a stiff string with an additional lumped mass can be calculated using Eq. (19). Instead of a representative value of B , practical values of a string used for middle C were chosen for the calculation. Those are shown in Table IV. The value B calculated from these parameters is 0.3636×10^{-3} which is fairly close to the value shown in reference [7]. The resonance frequencies of a stiff string with the F-F support condition were calculated for two values of mass ratio, $C = 0.1$ and 0.05 . For $C = 0.1$, the ratio of the distance of the mass from the edge of the string to the total length of the string ($\Delta l / l$) were chosen such that $\Delta l / l = 0.02, 0.015, 0.1, \text{ and } 0.0$. If $\Delta l / l = 0$, it is equivalent to the case with no additional mass. For $C = 0.05$, two values of ($\Delta l / l$) were selected for calculation such that $\Delta l / l = 0.02$ and 0.015 . The results are shown in Fig. 2 and Fig. 3 for $C = 0.1$ and 0.05 , respectively. The frequency deviations are shown by the inharmonicity defined below:

$$\text{Inharmonicity} = 1200 \log (f_n / n f_1) \text{ (cents)}$$

Obviously, a purely harmonic series has zero inharmonicity for any mode number n .

Curve (a) in Fig. 2 and 3 shows that the inharmonicity of a middle C string without an additional mass approaches approximately 120 cents (= semitone) for the 20th partial when the string is fixed-supported at both ends. For $C = 0.1$ and $\Delta\ell / \ell = 0.02$ (curve (d) in Fig. 2), the mass overcompensates the effect of the stiffness of the string and the inharmonicity becomes negative for mode numbers $n = 2 - 15$. In case of $C = 0.1$ and $\Delta\ell / \ell = 0.15$, the inharmonicity is very small for $n \leq 15$. Even for the higher mode numbers, the inharmonicity of this string is much smaller than that of a string without an additional mass. Curve (f) in Fig. 3 shows that even a mass having 5% of the string mass reduces the inharmonicity significantly when it is attached off the end of the string by 2% of its total length. Fig. 2 and Fig. 3 indicate that a proper combination of parameters around $C = 0.1 - 0.05$ and $\Delta\ell / \ell = 0.015 - 0.02$ may give a reasonable reduction of inharmonicity for the strings used around the key number 40 (middle C).

2.3 Comparison of Theoretical and Experimental Results

Figure 4 shows one end of an experimental setup used to measure the resonance frequencies of a string. The string is attached to a massive steel block through a holder and a force transducer. The holder gives the fixed condition to the string. The other end of the string is fixed by a holder alone. The transducer picks up the force caused by the vibration of the string. The length, the diameter, and the linear density of the string are those given in Table IV. The tuning screw enables one to adjust the tension of the string as desired. The fundamental resonance frequency was adjusted as close as possible to 261.6 Hz (middle C).

Then, the resonance frequencies up to the 20th mode were measured with and without the additional mass. The mass ratio C and the length ratio $\Delta l/l$ were 0.09 and 0.016, respectively.

The results are shown in Fig. 5 and Fig. 6 for the strings without and with the additional mass, respectively. The lines with circles are the theoretical results and the lines with Y are the experimental results. The theoretical and the experimental results of the string without the additional mass (Fig. 5) are in excellent agreement with each other. The results of the string with the additional mass (Fig. 6) are also in good agreement. The experimental results shows that the effect of the additional mass is slightly more than predicted. One reason considered for this is that the rotary inertia of the attached mass is neglected in the theory. It can be concluded, however, that the theory predicts the experimental results very well.

III CONCLUSION

At first, the theories to calculate the resonance frequencies of a stiff string with and without an additional mass were presented. The resonance frequencies of a stiff string for three representative values of B such as $B = 0.0001$, 0.001 , and 0.01 were calculated for three different support conditions, i.e., S-S, S-F, and F-F. The approximate formula to calculate the resonance frequencies of a stiff string given in reference [2] and [7] was found to be quite accurate for

low to mid values of B . The formula, however, does not hold for larger values of B (>0.01). In this region, the inharmonicity is largest for the S-S support and smallest for the F-F support. For smaller B , the inharmonicities are the same for the three support conditions.

The effects of the additional mass were calculated for a string with parameters corresponding to those of the key number 40 (middle C). An additional mass with a weight of 5 - 10% of the string weight, when attached at the place 2-1.5% of the total length of the string from one end, worked well to reduce the inharmonicity of the string. An experimental setup was made to measure the resonance frequencies of a real string with the F-F support condition. The agreement between the theoretical and experimental results was very good both for the strings with and without an additional mass.

As F. Miller, Jr. suggested⁶, it seems very important to make a complete piano with much less inharmonicity from the lowest to the highest key. This will provide us with precious information about how the inharmonicity affects the timber of each tone, the octave stretching, and the quality of a piano as a whole.

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CAPTIONS

- Table I Resonance frequencies of a stiff string with $B = 0.0001$ normalized to $f_0 = (T/\mu)^{1/2}/2\ell$.
- Table II Resonance frequencies of a stiff string with $B = 0.001$ normalized to $f_0 = (T/\mu)^{1/2}/2\ell$.
- Table III Resonance frequencies of a stiff string with $B = 0.01$ normalized to $f_0 = (T/\mu)^{1/2}/2\ell$.
- Table IV Parameters of a string used for middle C.
- Fig. 1 Fourth mode shape of a stiff string with $B = 0.01$ and with the F-F support condition.
- Fig. 2 Inharmonicities of a string with an additional mass. The curves a, b, c, and d are for $C = 0.1$ and $\Delta\ell/\ell = 0.0, 0.01, 0.015,$ and $0.02,$ respectively.
- Fig. 3 Inharmonicities of a string with an additional mass. The curves a, e, and f are for $C = 0.05,$ and $\Delta\ell/\ell = 0.0, 0.015,$ and $0.02,$ respectively.

Fig. 4 Experimental setup for the measurement of resonance frequencies of a string with F-F support condition.

Fig. 5 Theoretical (circle) and experimental (Y) inharmonicities of a string with an additional mass ($C = 0.09$ and $\Delta l/l = 0.16$). The parameters of the string are given in Table IV

Table I. Resonance frequencies of a stiff string with $B=0.0001$ normalized to $f_0 = (T/\mu)^{1/2}/2\ell$

MODE NUMBER	Boundary Conditions		
	S-S	S-F	F-F
1	1.00005	1.0032	1.0065
2	2.0004	2.0068	2.0132
3	3.0014	3.0109	3.0206
4	4.0032	4.0160	4.0289
5	5.0063	5.0222	5.0383
6	6.0108	6.0300	6.0493
7	7.0171	7.0396	7.0621
8	8.0256	8.0512	8.0770
9	9.0364	9.0653	9.0943
10	10.0499	10.0820	10.1144
11	11.0663	11.1018	11.1374
12	12.0861	12.1248	12.1637
13	13.1094	13.1514	13.1936
14	14.1365	14.1818	14.2274
15	15.1678	15.2164	15.2653
16	16.2035	16.2554	16.3077
17	17.2439	17.2992	17.3548
18	18.2893	18.3479	18.4070
19	19.3399	19.4020	19.4644
20	20.3961	20.4616	20.5274

Table II. Resonance frequencies of a stiff string with $B=0.001$ normalized to $f_0 = (T/\mu)^{1/2}/2\ell$

MODE NUMBER	Boundary Conditions		
	S-S	S-F	F-F
1	1.0005	1.0107	1.0211
2	2.0040	2.0244	2.0452
3	3.0135	3.0442	3.0755
4	4.0319	4.0730	4.1149
5	5.0621	5.1138	5.1665
6	6.1070	6.1694	6.2331
7	7.1695	7.2428	7.3176
8	8.2520	8.3366	8.4227
9	9.3574	9.4534	9.5511
10	10.4881	10.5957	10.7053
11	11.6465	11.7661	11.8879
12	12.8350	12.9668	13.1010
13	14.0556	14.2000	14.3469
14	15.3106	15.4678	15.6276
15	16.6020	16.7722	16.9452
16	17.9314	18.1150	18.3015
17	19.3008	19.4980	19.6982
18	20.7117	20.9227	21.1369
19	22.1658	22.3909	22.6192
20	23.6643	23.9037	24.1465

Table III. Resonance frequencies of a stiff string with $B=0.01$ normalized to $f_0 = (T/\mu)^{1/2}/2\ell$

MODE NUMBER	Boundary Conditions		
	S-S	S-F	F-F
1	1.0050	1.0381	1.0734
2	2.0396	2.1069	2.1786
3	3.1321	3.2358	3.3458
4	4.3081	4.4507	4.6014
5	5.5902	5.7744	5.9681
6	6.9971	7.2254	7.4644
7	8.5446	8.8191	9.1053
8	10.2450	10.5675	10.9023
9	12.1083	12.4801	12.8643
10	14.1421	14.5643	14.9996
11	16.3527	16.8258	17.3124
12	18.7446	19.2693	19.8075
13	21.3216	21.8981	22.4883
14	24.0865	24.7151	25.3574
15	27.0416	27.7224	28.4169
16	30.1887	30.9281	31.6685
17	33.5293	34.3145	35.1134
18	37.0643	37.9017	38.7528
19	40.7947	41.6843	42.5876
20	44.7214	45.6631	46.6183

TABLE IV. Parameters of a string used for middle C.

Young's Modulus	E	19.5×10^{11} dyne/cm ²
Diameter	D	0.1042 cm
Linear Density	μ	0.0672 gr/cm
Length	ℓ	63.88 cm
Resonance Frequency	f_0	261.6 Hz

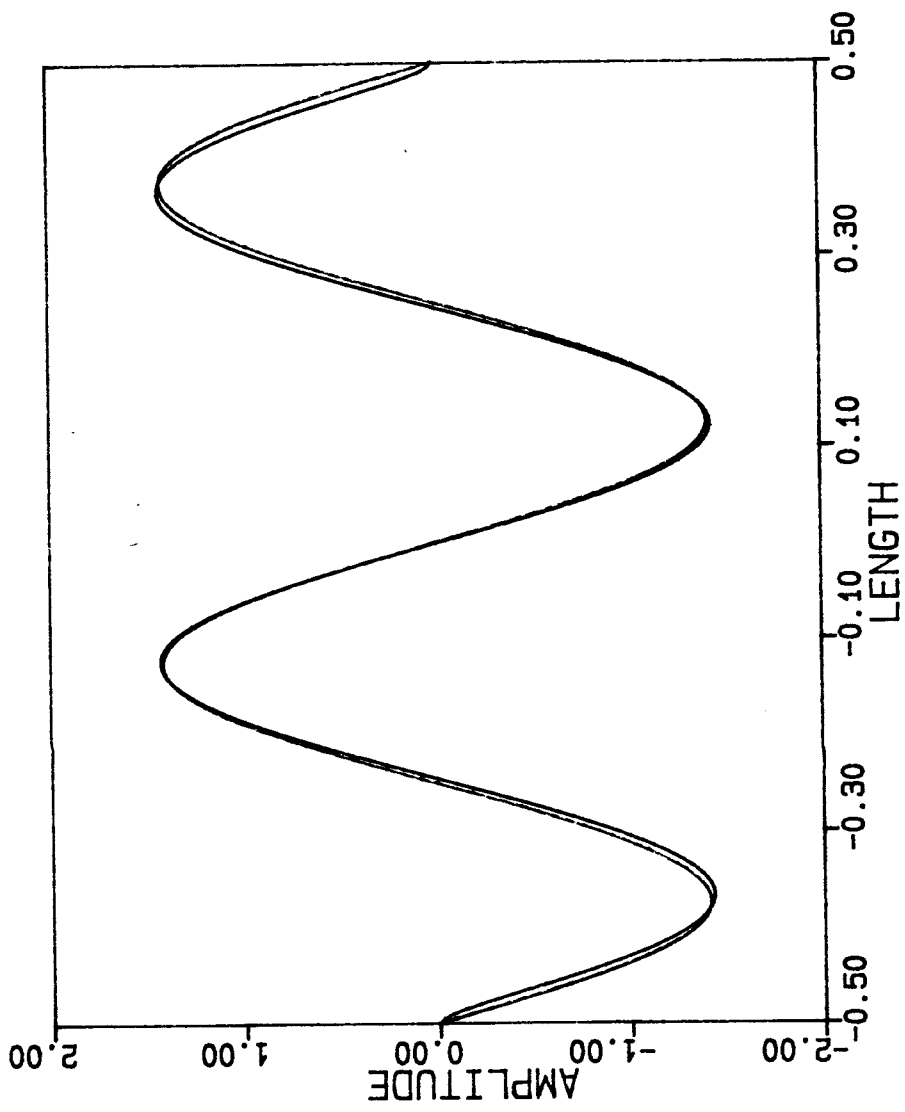


Fig. 1 Fourth eigenfunction of a stiff string with $B = 0.01$ and with the F-F support.

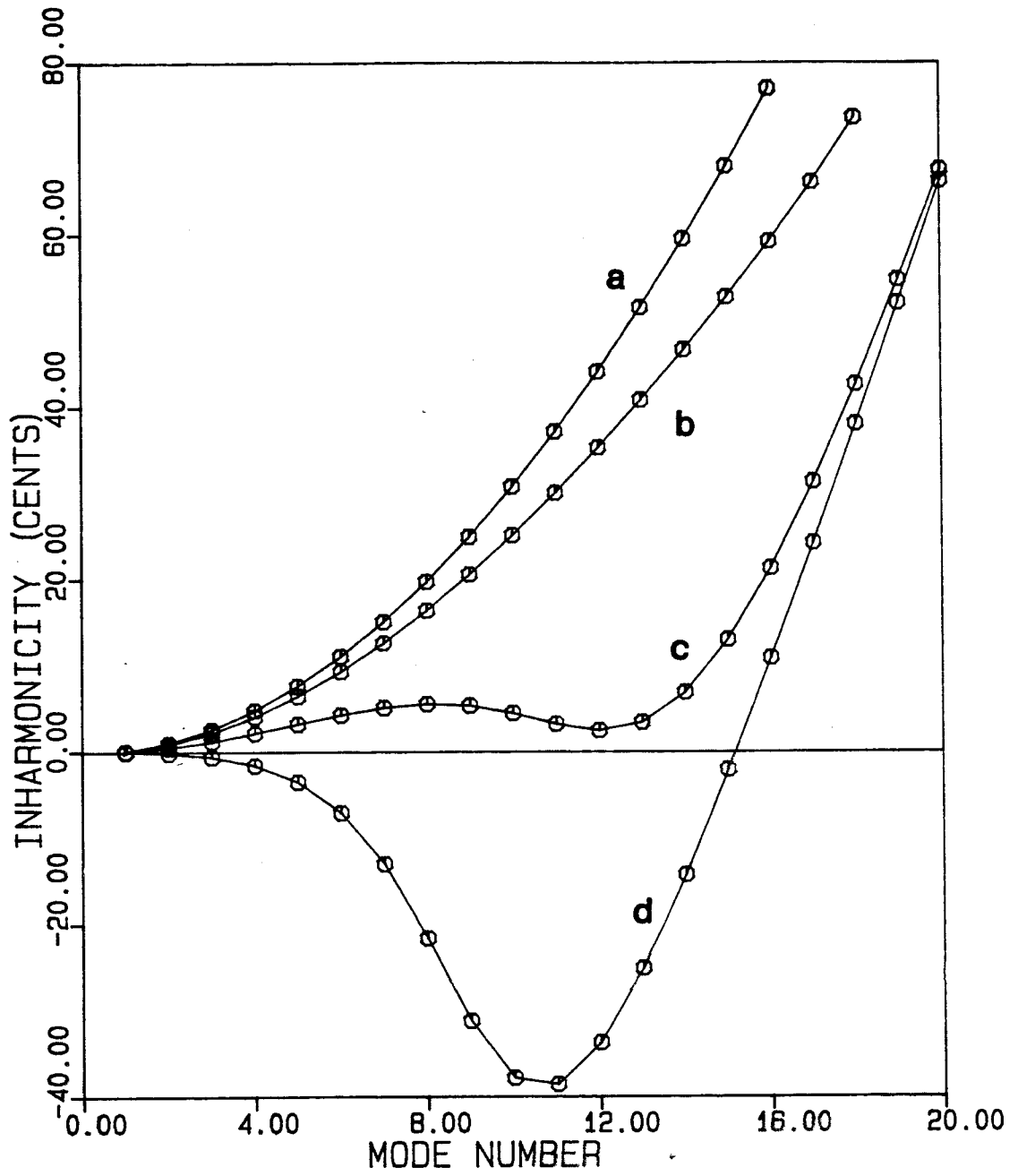


Fig. 2 Inharmonicities of a string with an additional mass. The curves a, b, c, and d are for $C + 0.1$, and $\Delta l/l = 0.0, 0.01, 0.015, \text{ and } 0.02$, respectively.

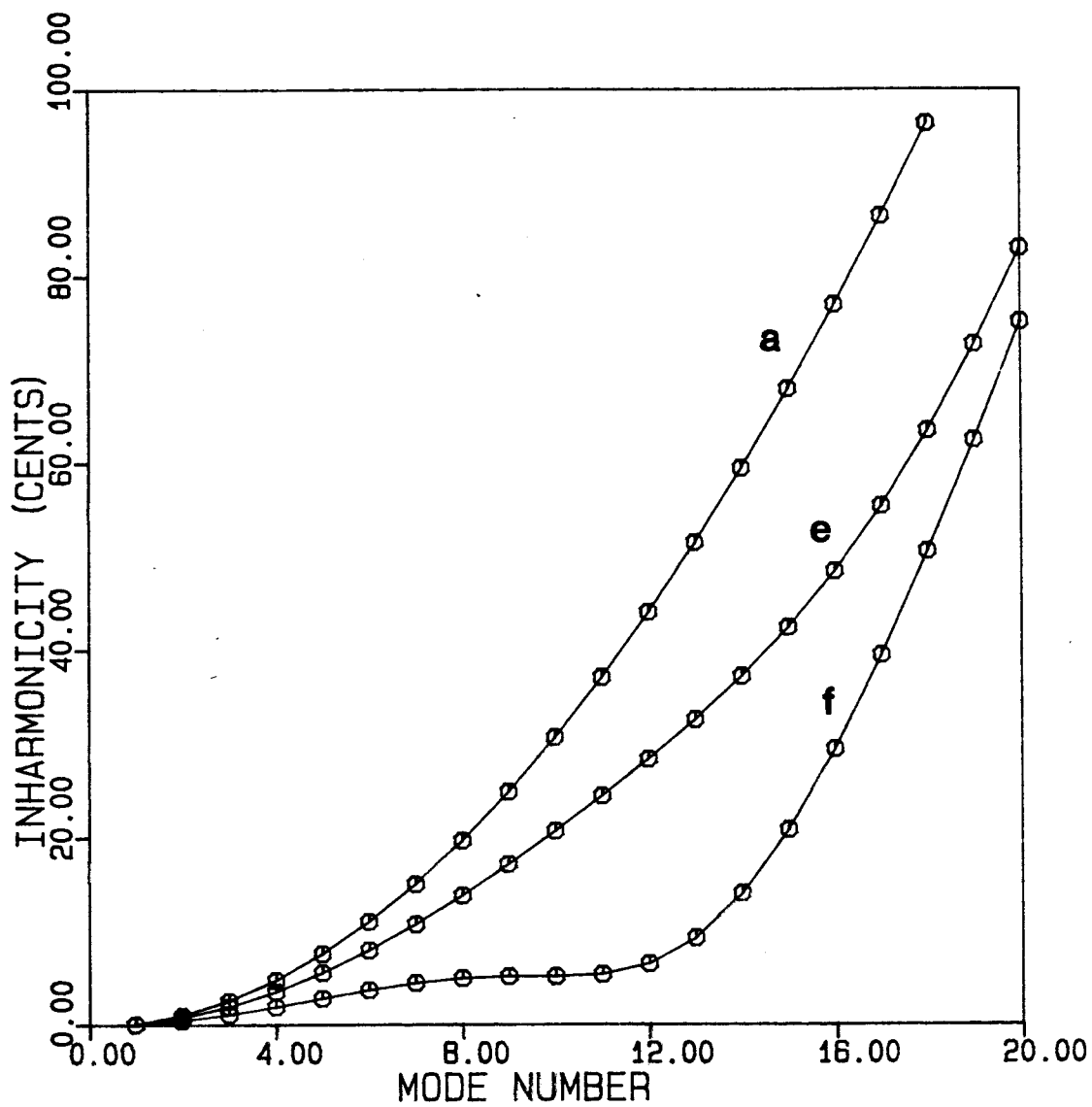


Fig. 3 Inharmonicities of a string with an additional mass. The curves a, e, and f are $C = 0.05$ and $\Delta l/l = 0.0, 0.015,$ and $0.02,$ respectively.

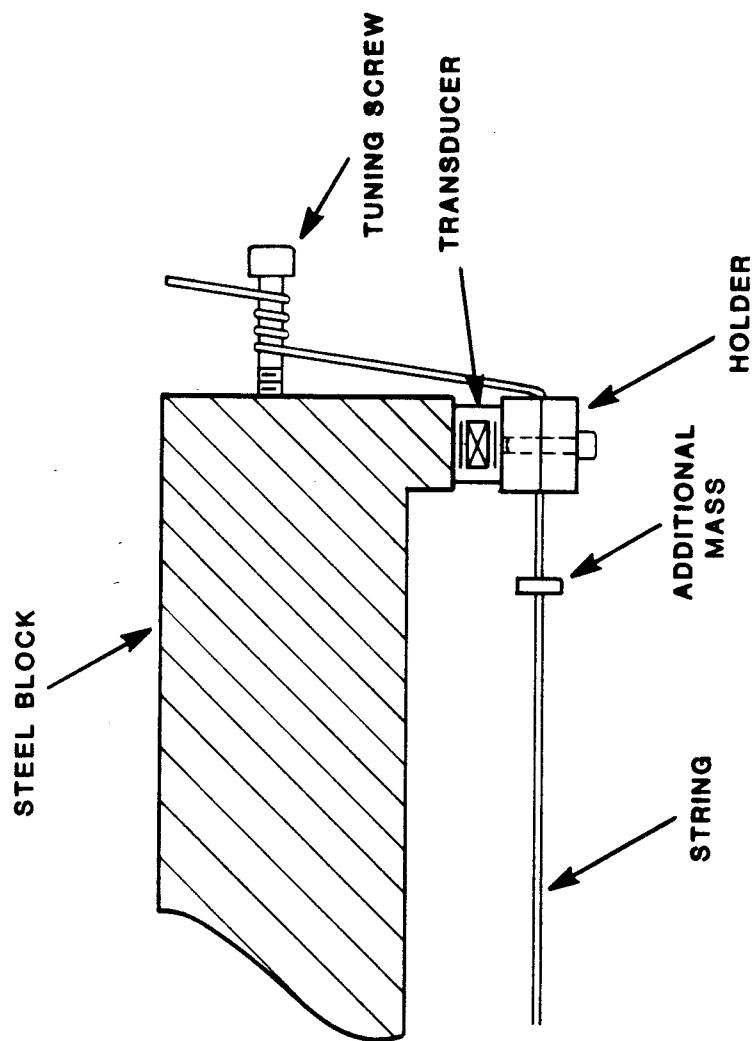


Fig. 4 Experimental setup for the measurement of resonance frequencies of a string with F-F support condition.

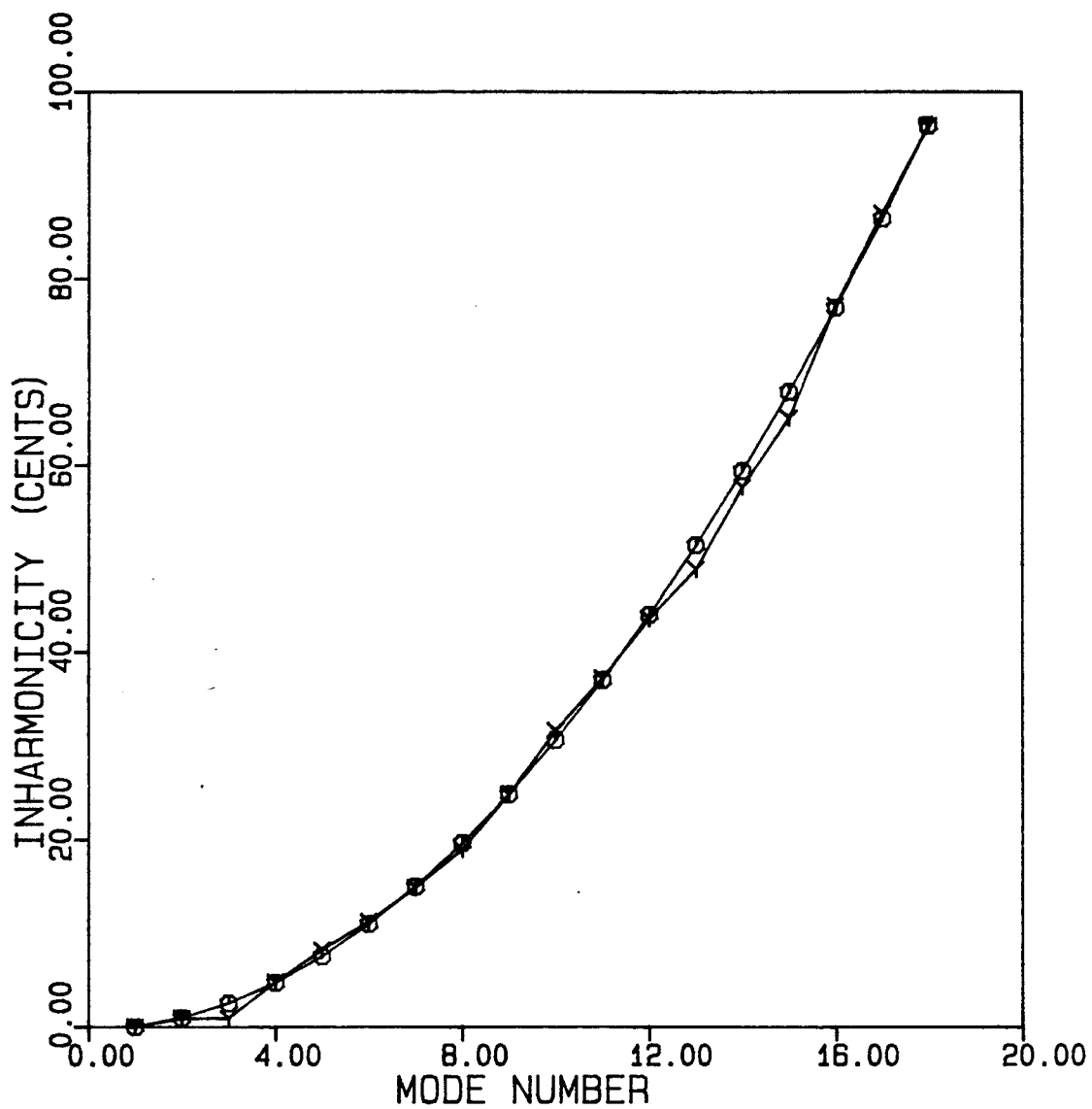


Fig. 5 Theoretical (circle) and experimental (Y) inharmonicities of a stiff string without an additional mass. The parameters of the string are given in Table IV.

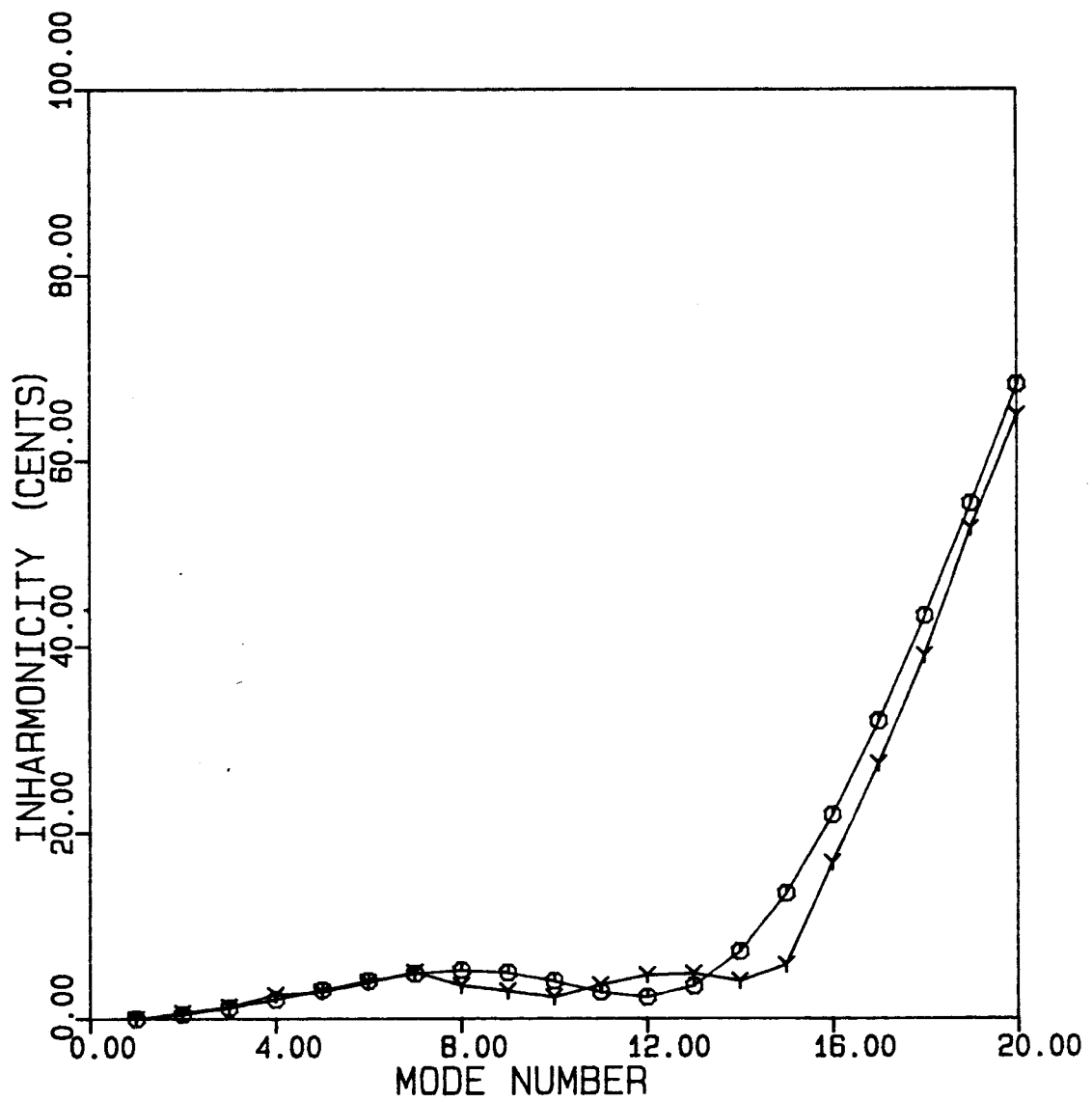


Fig. 6 Theoretical (circle) and experimental (Y) inharmonicities of a string with an additional mass ($C = 0.09$ and $\Delta\ell/\ell = 0.16$). The parameters of the string are given in Table IV.